ENS ICFP MASTER - First year - 2018/2019
Relativistic quantum mechanics and introduction to quantum field theory
Solution of the homework

1 Some operator identities

Note first that if $O(t)$ is a t-dependent operator whose derivative $\frac{\mathrm{d}O}{\mathrm{d}t}$ commutes with $O(t)$, then $\frac{\mathrm{d}}{\mathrm{d}t}e^{O(t)} = \frac{\mathrm{d}O}{\mathrm{d}t}e^{O(t)} = e^{O(t)}\frac{\mathrm{d}O}{\mathrm{d}t}$, as can be proven by deriving term by term the series defining the exponential.

a) With $O(t) = tA$, one has $\frac{\mathrm{d}O}{\mathrm{d}t} = A$ that commutes with $O(t)$, hence

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}Xe^{-tA}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right)Xe^{-tA} + e^{tA}X\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tA}\right) = e^{tA}(AX - XA)e^{-tA} \\ &= e^{tA}[A,X]e^{-tA} \end{aligned}$$

for any operator X. By induction on n one deduces that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}F(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(e^{tA}Be^{-tA}\right) = e^{tA}[A,[A,\dots,[A,B]\dots]]e^{-tA}$$

with n commutators. The identity (1) of the problem then follows from

$$F(1) = F(0) + \sum_{n=1}^{\infty}\frac{1}{n!}\left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}F(t)\right|_{t=0} \ ;$$

the Taylor expansion of $F(t)$ in 0 has indeed an infinite radius of convergence for bounded operators A.

b) One computes the derivative of $G(t)$ as suggested,

$$\begin{aligned} \frac{\mathrm{d}G}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}e^{tB}\right) = Ae^{tA}e^{tB} + e^{tA}Be^{tB} = Ae^{tA}e^{tB} + \left(e^{tA}Be^{-tA}\right)e^{tA}e^{tB} \\ &= Ae^{tA}e^{tB} + (B + t[A,B])\,e^{tA}e^{tB}\ , \end{aligned}$$

where in the last step we used the identity (1), the series stopping at $n = 1$ because A commutes with $[A,B]$. The differential equation thus obtained, $\frac{\mathrm{d}G}{\mathrm{d}t} = (A + B + t[A,B])G(t)$, with $G(0) = \mathbf{1}$, can be integrated in $G(t) = \exp\left[t(A+B) + \frac{t^2}{2}[A,B]\right]$ thanks to the preliminary remark above and the commutation of $A + B$ with $[A,B]$. The identity (2) then follows with $t = 1$.

c) Taking $A = \int \mathrm{d}\vec{q}\,g(\vec{q})a^\dagger(\vec{q})$ and $B = \int \mathrm{d}\vec{q}\,f(\vec{q})a(\vec{q})$, one has $[A,B] = \int \mathrm{d}\vec{q_1}\mathrm{d}\vec{q_2}\,g(\vec{q_1})f(\vec{q_2})[a^\dagger(\vec{q_1}),a(\vec{q_2})] = -\int \mathrm{d}\vec{q}\,g(\vec{q})f(\vec{q})$; then (3) follows directly from (2).

2 Some Lorentz algebra

We shall use the identity (1) with $A = -\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}$ and $B = P^\mu$. We compute the commutator

$$[A,B] = -\frac{i}{2}\omega_{\rho\sigma}[J^{\rho\sigma},P^\mu] = -\frac{i}{2}\omega_{\rho\sigma}(-i)(\eta^{\sigma\mu}P^\rho - \eta^{\rho\mu}P^\sigma) = \omega_{\rho\sigma}\eta^{\rho\mu}P^\sigma = \omega^\mu{}_\sigma P^\sigma\ ,$$

where we used (3.50) from the lecture notes and the antisymmetry of $\omega_{\rho\sigma}$. We have thus

$$[A,P^\mu] = \omega^\mu{}_\rho\,P^\rho\ , \qquad \text{hence} \quad [A,[A,\dots[A,P^\mu]\dots]] = (\omega^n)^\mu{}_\rho P^\rho$$

with n commutators on the left-hand-side and ω^n being the n-th matrix power of ω. The identity (5) is then obtained by resumming the series in (1) to obtain $e^\omega = \Lambda$.

3 Relations for products of γ-matrices and their traces

The Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$ translates into $(\gamma^0)^2 = -\mathbf{1}$, $(\gamma^i)^2 = \mathbf{1}$ for $i = 1, 2, 3$, and the anti-commutation of γ^μ and γ^ν if $\mu \neq \nu$. The matrices with covariant indices are defined as usual by $\gamma_0 = -\gamma^0$, $\gamma_i = \gamma^i$ for $i = 1, 2, 3$.

$\gamma_\mu\gamma^\mu = -(\gamma^0)^2 + (\gamma^1)^2 + (\gamma^2)^2 + (\gamma^3)^2 = 4 \times \mathbf{1}$

$\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu(\{\gamma^\nu, \gamma^\mu\} - \gamma^\mu\gamma^\nu) = 2\eta^{\mu\nu}\gamma_\mu - \gamma_\mu\gamma^\mu\gamma^\nu = -2\gamma^\nu$, by using the previous result.

$\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu = \gamma_\mu\gamma^\nu(\{\gamma^\rho, \gamma^\mu\} - \gamma^\mu\gamma^\rho) = 2\eta^{\rho\mu}\gamma_\mu\gamma^\nu - \gamma_\mu\gamma^\nu\gamma^\mu\gamma^\rho = 2\gamma^\rho\gamma^\nu + 2\gamma^\nu\gamma^\rho = 2\{\gamma^\rho, \gamma^\nu\} = 4\eta^{\rho\nu} \times \mathbf{1}$

$\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu = \gamma_\mu\gamma^\nu\gamma^\rho(\{\gamma^\sigma, \gamma^\mu\} - \gamma^\mu\gamma^\sigma) = 2\eta^{\sigma\mu}\gamma_\mu\gamma^\nu\gamma^\rho - 4\eta^{\rho\nu}\gamma^\sigma = 2\gamma^\sigma(\gamma^\nu\gamma^\rho - \{\gamma^\nu, \gamma^\rho\}) = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

$\mathrm{tr}\,\gamma_\mu\gamma_\nu = \frac{1}{2}\mathrm{tr}\,(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu}\mathrm{tr}\,\mathbf{1} = 4\eta_{\mu\nu}$

In the following proof we move γ_μ to the right by using the Clifford algebra :

$$\begin{aligned}
\mathrm{tr}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma &= 2\eta_{\mu\nu}\mathrm{tr}\,\gamma_\rho\gamma_\sigma - \mathrm{tr}\,\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma \\
&= 2\eta_{\mu\nu}\mathrm{tr}\,\gamma_\rho\gamma_\sigma - 2\eta_{\mu\rho}\mathrm{tr}\,\gamma_\nu\gamma_\sigma + \mathrm{tr}\,\gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma \\
&= 2\eta_{\mu\nu}\mathrm{tr}\,\gamma_\rho\gamma_\sigma - 2\eta_{\mu\rho}\mathrm{tr}\,\gamma_\nu\gamma_\sigma + 2\eta_{\mu\sigma}\mathrm{tr}\,\gamma_\nu\gamma_\rho - \mathrm{tr}\,\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu
\end{aligned}$$

By the cyclicity of the trace the last term on the right is equal to the term on the left hand side, hence $\mathrm{tr}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = \frac{1}{2}(2\eta_{\mu\nu}\mathrm{tr}\,\gamma_\rho\gamma_\sigma - 2\eta_{\mu\rho}\mathrm{tr}\,\gamma_\nu\gamma_\sigma + 2\eta_{\mu\sigma}\mathrm{tr}\,\gamma_\nu\gamma_\rho) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$

We recall that $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ is a matrix that anticommutes with the four γ^μ's, and that square to one. Thus

$$\mathrm{tr}\,\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}} = \mathrm{tr}\,\gamma_5\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}} = \begin{cases} \mathrm{tr}\,\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}\gamma_5 \\ -\mathrm{tr}\,\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}\gamma_5 \end{cases},$$

where in the first line we used the cyclicity of the trace, and in the second line the fact that γ_5 anticommutes with the product of an odd number of γ^μ's. This implies that $\mathrm{tr}\,\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}} = 0$.

4 Energy levels of a relativistic charged spin-0 particle in a constant magnetic field

a) With $\vec{A} = A_z(x, y, z)\vec{e}_z$ the associated magnetic field is $\vec{B} = \vec{\nabla} \wedge \vec{A} = (\partial_y A_z)\vec{e}_x - (\partial_x A_z)\vec{e}_y$, we can thus take $A_z = -Bx$ to have $\vec{B} = B\vec{e}_y$. The Klein-Gordon equation in presence of an electromagnetic potential is

$$\left[-(\partial_\mu - iqA_\mu)(\partial^\mu - iqA^\mu) + m^2\right]\phi = 0 \ .$$

With the above choice for $\vec{A}$ one obtains

$$\left[\frac{\partial^2}{\partial t^2} - \Delta + m^2 + q^2B^2x^2 - 2iqBx\frac{\partial}{\partial z}\right]\phi(t, \vec{x}) = 0 \ ,$$

with Δ the spatial Laplacian.

b) We shall look for a stationary solution of the form $\phi(t, \vec{x}) = e^{-iEt}\varphi(\vec{x})$, where $\varphi(\vec{x})$ obeys

$$\left[-E^2 - \Delta + m^2 + q^2B^2x^2 - 2iqBx\frac{\partial}{\partial z}\right]\varphi(\vec{x}) = 0 \ .$$

Denoting the momentum operator $\vec{P} = -i\vec{\nabla}$ this can be rewritten as an eigenvalue problem,

$$\left[P_x^2 + P_y^2 + (P_z + qBX)^2\right]\varphi = (E^2 - m^2)\varphi.$$

As the operator in the left hand side commutes with P_y and P_z we can look for a common eigenvector, i.e. take $\varphi(x, y, z) = e^{ik_y y + ik_z z}\chi(x)$, with

$$\left[P_x^2 + (qB)^2\left(X + \frac{k_z}{qB}\right)^2\right]\chi = (E^2 - m^2 - k_y^2)\chi \ .$$

Dividing by $2m$ we obtain

$$\left[\frac{P_x^2}{2m}+\frac{1}{2}m\left(\frac{qB}{m}\right)^2\left(X+\frac{k_z}{qB}\right)^2\right]\chi=\frac{E^2-m^2-k_y^2}{2m}\chi\ .$$

One recognizes a Schrödinger equation for an harmonic oscillator of mass m and frequency $\omega = qB/m$, the location of the center $x_0 = -k_z/(qB)$ of the harmonic well being fixed by k_z. The eigenvalues of this problem are $\omega(n + (1/2))$, with $n = 0, 1, \dots,$ hence

$$E(n,k_y)=\sqrt{m^2+k_y^2+qB(2n+1)}\ ,\qquad \varphi(\vec{x})=e^{ik_yy+ik_zz}e^{-\frac{qB}{2}(x+(k_z/qB))^2}H_n(\sqrt{qB}(x+(k_z/qB)))\ ,$$

with H_n the Hermite polynomials. For each value of n and k_y there is an infinite degeneracy due to the free choice of k_z.

c) Expanding in the large m limit the above expression yields

$$E(n,k_y)=m+\left[\frac{k_y^2}{2m}+\frac{qB}{m}\left(n+\frac{1}{2}\right)\right]-\frac{1}{8m^3}\left(k_y^2+2qB\left(n+\frac{1}{2}\right)\right)^2+\dots$$

The first term is the rest mass energy, the square bracket is the non-relativistic result that would have been obtained with the Schrödinger equation, the last term is the first relativistic correction.

d) The magnetic field is invariant under the transformation $\vec{A}\to\vec{A}+\vec{\nabla}f$, for an arbitrary function $f(\vec{x})$. The Klein-Gordon equation is invariant if one performs simultaneously the gauge transformation $\phi \to \phi e^{iqf}$, hence a different choice of vector potential would only add a (space-dependent) phase to the wavefunctions, but does not change the energies of the Landau levels.

5 Weakly relativistic limit of the Dirac equation and spin-orbit coupling

a) In the Dirac representation

$$\gamma^0=-i\begin{pmatrix}\mathrm{Id} & 0\\ 0 & -\mathrm{Id}\end{pmatrix}\ ,\qquad \gamma^j=-i\begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}\ .$$

The Dirac equation in presence of an electrostatic potential $V(\vec{x})$ reads

$$\left(\gamma^0\left(\frac{\partial}{\partial t}+iqV(\vec{x})\right)+\vec{\gamma}\cdot\vec{\nabla}+m\right)\psi(t,\vec{x})=0\ .$$

For a wave-function of the form proposed in the text, and using the Dirac representation of the gamma matrices, one obtains the following coupled equations for φ and χ :

$$\begin{cases}(-\epsilon+qV(\vec{x})\varphi(\vec{x})-i(\vec{\sigma}\cdot\vec{\nabla})\chi(\vec{x})=0\\ (2m+\epsilon-qV(\vec{x}))\chi(\vec{x})+i(\vec{\sigma}\cdot\vec{\nabla})\varphi(\vec{x})=0\end{cases}$$

b) With the second equation one gets χ in terms of φ as

$$\chi(\vec{x})=\frac{1}{2m+\epsilon-qV(\vec{x})}(-i\vec{\sigma}\cdot\vec{\nabla})\varphi(\vec{x})\ .$$

Reinserting in the first equation yields

$$\epsilon\varphi(\vec{x})=\left[qV(\vec{x})+(-i\vec{\sigma}\cdot\vec{\nabla})\frac{1}{2m+\epsilon-qV(\vec{x})}(-i\vec{\sigma}\cdot\vec{\nabla})\right]\varphi(\vec{x})\ ,$$

i.e. the form of the text with

$$f(\vec{x}) = qV(\vec{x}) \ , \qquad g(\vec{x}) = \frac{1}{2m + \epsilon - qV(\vec{x})} \ .$$

c) Expanding at large m one has

$$g(\vec{x}) = \frac{1}{2m}\frac{1}{1 + \frac{\epsilon - qV(\vec{x})}{2m}} = \frac{1}{2m} - \frac{\epsilon - qV(\vec{x})}{4m^2} + O\left(\frac{\epsilon^2}{m^3}\right) \ ,$$

which yields equation (15) of the text.

d) $(\vec{\sigma}\cdot\vec{P})^2 = \sigma_i\sigma_j P_i P_j = (\delta_{ij} + i\epsilon_{ijk}\sigma_k)P_iP_j = \vec{P}^2$ as P_iP_j is symmetric and ϵ_{ijk} antisymmetric under the exchange $i \leftrightarrow j$. Moreover

$$[\vec{\sigma}\cdot\vec{P}, \epsilon - qV(\vec{x})] = -q\vec{\sigma}\cdot[\vec{P}, V(\vec{x})] = iq\vec{\sigma}\cdot(\vec{\nabla}V) \ .$$

Hence

$$\begin{aligned} H_P &= \frac{\vec{P}^2}{2m} + qV(\vec{x}) - \frac{1}{4m^2}\left[(\vec{\sigma}\cdot\vec{P})(\vec{\sigma}\cdot\vec{P})(\epsilon - qV(\vec{x})) - (\vec{\sigma}\cdot\vec{P})[\vec{\sigma}\cdot\vec{P}, \epsilon - qV(\vec{x})]\right] \\ &= \frac{\vec{P}^2}{2m} + qV(\vec{x}) - \frac{1}{4m^2}\vec{P}^2(\epsilon - qV(\vec{x})) + \frac{iq}{4m^2}(\vec{\sigma}\cdot\vec{P})(\vec{\sigma}\cdot(\vec{\nabla}V)) \end{aligned}$$

The last term is the spin-orbit coupling Hamiltonian ; thanks to the properties of multiplication of the Pauli matrices one has

$$(\vec{\sigma}\cdot\vec{P})(\vec{\sigma}\cdot(\vec{\nabla}V)) = \vec{P}\cdot(\vec{\nabla}V) + i\vec{\sigma}\cdot(\vec{P}\wedge(\vec{\nabla}V))$$

For a spherically symmetric potential $\vec{\nabla}V = \vec{r}\frac{V'(r)}{r}$, which explains the second term in (17) with $b(r) = qV'(r)/(4r)$. Moreover one can check (by applying these operators to arbitrary test functions) that for any spherically symmetric function $f(r)$ one has $\vec{P}\wedge f(r)\vec{r} = f(r)\vec{r}\wedge\vec{P} = f(r)\vec{L}$, hence the first term in (17) with $a(r) = -qV'(r)/(2r)$, the spin operator being $\vec{S} = \vec{\sigma}/2$.

e) At the lowest order the wave function φ obeys $(\epsilon - qV)\varphi = \frac{\vec{P}^2}{2m}\varphi$, one can thus replace in the correction term of H_P

$$-\frac{1}{4m^2}\vec{P}^2(\epsilon - qV(\vec{x})) = -\frac{(\vec{P}^2)^2}{8m^3} \ .$$

6 The axial current

a) As γ_5 anticommutes with the γ^μ's, $(\gamma_5)^p\gamma^\mu = (-1)^p\gamma^\mu(\gamma_5)^p$ for integer values of p ; expanding the exponential in series then shows the identity $e^{i\epsilon\gamma_5}\gamma^\mu = \gamma^\mu e^{-i\epsilon\gamma_5}$. As γ_5 is Hermitian,

$$\overline{\psi}(x) \to \left(e^{i\epsilon\gamma_5}\psi\right)^\dagger i\gamma^0 = \psi^\dagger e^{-i\epsilon\gamma_5} i\gamma^0 = \psi^\dagger i\gamma^0 e^{i\epsilon\gamma_5} = \overline{\psi}(x)e^{i\epsilon\gamma_5}$$

b) Indeed, $\overline{\psi}\psi$ is not invariant under this transformation, whereas $\overline{\psi}\gamma^\mu\psi$ is :

$$\begin{aligned} &\overline{\psi}\psi \to \overline{\psi}e^{2i\epsilon\gamma_5}\psi \neq \overline{\psi}\psi \\ &\overline{\psi}\gamma^\mu\psi \to \overline{\psi}e^{i\epsilon\gamma_5}\gamma^\mu e^{i\epsilon\gamma_5}\psi = \overline{\psi}\gamma^\mu e^{-i\epsilon\gamma_5}e^{i\epsilon\gamma_5}\psi = \overline{\psi}\gamma^\mu\psi \end{aligned}$$

Consider the variation of the massless action under an infinitesimal transformation with $\delta\psi = i\epsilon\gamma_5\psi$, $\delta\overline{\psi} = i\epsilon\overline{\psi}\gamma_5$, where ϵ is space dependent :

$$\delta S = \int \mathrm{d}^4x \left[\overline{\psi}(-\gamma^\mu\partial_\mu + iq\gamma^\mu A_\mu)i\epsilon\gamma_5\psi + i\epsilon\overline{\psi}\gamma_5(-\gamma^\mu\partial_\mu + iq\gamma^\mu A_\mu)\psi\right] = \int \mathrm{d}^4x \left[-i(\partial_\mu\epsilon)\overline{\psi}\gamma^\mu\gamma_5\psi\right] \ ,$$

the other terms vanishing because γ_5 anticommutes with the γ^μ's. Integrating by part we see that when the Euler-Lagrange equations are verified $\partial_\mu j_5^\mu = 0$, i.e. the axial current is conserved for a massless Dirac field.

c) The Dirac equations for ψ and $\overline{\psi}$ are

$$(\not\partial - iq\not A + m)\psi = 0, \qquad \overline{\psi}(\overleftarrow{\not\partial} + iq\not A - m) = 0 \ ,$$

where the derivatives in $\overleftarrow{\not\partial}$ acts on the left. Hence

$$\begin{aligned} \partial_\mu j_5^\mu &= i\overline{\psi}\gamma_5\gamma^\mu\partial_\mu\psi + i(\partial_\mu\overline{\psi})\gamma_5\gamma^\mu\psi = i\overline{\psi}\gamma_5\not\partial\psi - i\overline{\psi}\overleftarrow{\not\partial}\gamma_5\psi = i\overline{\psi}\left[\gamma_5(iq\not A - m) - (-iq\not A + m)\gamma_5\right]\psi \\ &= -2im\overline{\psi}\gamma_5\psi \end{aligned}$$

as $\not A\gamma_5 = -\gamma_5\not A$. We find again that j_5^μ is conserved if and only if $m = 0$.

d) Taking the complex conjugate of the μ-th component of j_5 we obtain

$$(j_5^\mu)^* = -i(\psi^\dagger i\gamma^0\gamma_5\gamma^\mu\psi)^\dagger = i\psi^\dagger(\gamma^\mu)^\dagger(\gamma_5)^\dagger i(\gamma^0)^\dagger\psi = i\psi^\dagger i\gamma^0\gamma^\mu\gamma^0\gamma_5(-\gamma^0)\psi = j_5^\mu \ ,$$

which is thus real. Under a Lorentz transformation $x \to x' = \Lambda x$ one has

$$\begin{cases} \psi'(x') = D(\Lambda)\psi(x) \\ \overline{\psi}'(x') = \overline{\psi}(x)D(\Lambda)^{-1} \end{cases} \qquad \Rightarrow \qquad (j_5^\mu)'(x') = i\overline{\psi}(x)D(\Lambda)^{-1}\gamma_5\gamma^\mu D(\Lambda)\psi(x) \ ,$$

where $D(\Lambda) = \exp[\frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu}]$ expands into terms that contain an even number of γ^μ matrices, hence commutes with γ_5. As $D(\Lambda)^{-1}\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$, one has $(j_5^\mu)'(x') = \Lambda^\mu{}_\nu j_5^\nu(x)$, i.e. j_5 transforms as a four-vector under Lorentz transformations. Under the parity transformation,

$$\begin{cases} \psi'(x') = i\gamma^0\psi(x) \\ \overline{\psi}'(x') = \overline{\psi}(x)i\gamma^0 \end{cases} \qquad \Rightarrow \qquad (j_5^\mu)'(x') = i\overline{\psi}(x)i\gamma^0\gamma_5\gamma^\mu i\gamma^0\psi(x) = i\overline{\psi}(x)\gamma_5\gamma^0\gamma^\mu\gamma^0\psi(x) \ .$$

As

$$\gamma^0\gamma^\mu\gamma^0 = \begin{cases} -\gamma^\mu & \text{if } \ \mu = 0 \\ +\gamma^\mu & \text{if } \ \mu = 1,2,3 \end{cases} \ , \qquad (1)$$

one obtains that j_5 is a pseudo-vector.