ICFP Masters program
École Normale Supérieure

M2 *(Fall/Winter 2018)*

Final Exam: Quantum Field Theory

This exam is closed book, closed notes. It consists of seven pages. Useful formulae are collected in appendix A and the Feynman rules of QED are summarized in appendix B. It is essential that you exhibit your calculations with all required intermediate steps. You have 3 hours. Good luck!

1 Pseudoscalar decay to photons

Consider the Lagrangian density

$$\mathcal{L}_\pi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\pi(\Box + m_\pi^2)\pi + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi + i\lambda\,\pi\bar{\psi}\gamma^5\psi\,. \tag{1}$$

This Lagrangian describes QED coupled to a (one-component) field π.

1. The Feynman rules for renormalized QED are given in appendix B. In this problem, we do not need to worry about renormalization, so we will instead consider the Feynman rules for the bare Lagrangian. Write down these rules for $\mathcal{L}_\pi$.

2. Recall the transformation of Dirac fermions in the Weyl basis under parity:
$$P\psi(x)P^{-1} = \gamma^0\psi(\mathcal{P}x)\,. \tag{2}$$
We have denoted the matrix implementing the parity transformation on $\mathbb{R}^{1,3}$ by $\mathcal{P}$. How must the field π transform under parity in order for the action of the theory which is defined by $\mathcal{L}_\pi$ to be parity invariant?

3. Draw the lowest order Feynman diagrams contributing to the decay $\pi \to \gamma\gamma$, where we have indicated photons by the letter γ. Designate by the letter p the momentum carried by π, and the momenta of the two photons by the letters q_1 and q_2. Label all internal lines with the momentum flowing through them.

4. Write down the matrix element $i\mathcal{M}$ corresponding to the contribution from these Feynman diagrams. Your result should be of the form
$$i\mathcal{M} = C_{\mu\nu}M^{\mu\nu}(q_1, q_2)\,, \tag{3}$$
with $C_{\mu\nu}$ depending on the coupling constants and polarization tensors, and with $M^{\mu\nu}$ of the form
$$\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\ldots. \tag{4}$$
Leave the momenta q_1 and q_2 general until further notice.

5. What is the superficial degree of divergence of $i\mathcal{M}$?

6. How does $i\mathcal{M}$ behave under exchange of the two photons? Use this behavior together with Lorentz invariance to constrain the dependence of $M^{\mu\nu}$ on q_1^μ and q_2^ν. You will need to use the fully antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$. Does this dependence allow you to modify your prediction regarding the convergence or divergence of the d^4k integral in $M^{\mu\nu}$?

7. Evaluate the trace in $M^{\mu\nu}$. You should obtain a result proportional to

$$M^{\mu\nu} = 8m\,\epsilon^{\mu\nu\alpha\beta} q_\alpha^1 q_\beta^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k-q_1)^2 - m^2]\,[(k+q_2)^2 - m^2]\,[k^2 - m^2]}\,. \tag{5}$$

8. Rewrite $M^{\mu\nu}$ using Feynman parameters. Express the result in terms of the momentum variables q_1^2, q_2^2, and $s = (q_1 + q_2)^2$.

9. Take all momentum variables on-shell, and evaluate the integrals over the Feynman parameters in the limit $m_\pi \ll m$. What is the final result for $\mathcal{M}$ in this limit?

2 Time reversal

Recall that the time reversal operator $\mathcal{T}$,

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

is represented on Hilbert space by an anti-unitary operator T that satisfies

$$TU(\Lambda, a)T^{-1} = U(\mathcal{T}\Lambda\mathcal{T}^{-1}, \mathcal{T}a) \tag{7}$$

for Λ a proper orthochronous Lorentz transformation, $\Lambda \in SO^\uparrow(1,3)$, and for $a \in \mathbb{R}^4$.

1. Work out the action of T on the Lie algebra generators $J^{\mu\nu}$ of Lorentz transformations and the Lie algebra generators P^μ of space-time translations. Next, write the action explicitly on the Hamiltonian, the momentum operators, the rotation operators, and the boost operators.

2. Why was it necessary to choose T to be anti-unitary?

3. Consider a massive eigenstate $|\psi_{k,\sigma}\rangle$ of the space-time translation and rotation group, where $k^\mu = (M, 0, 0, 0)$,

$$P^\mu|\psi_{k,\sigma}\rangle = k^\mu|\psi_{k,\sigma}\rangle\,, \quad J_3|\psi_{k,\sigma}\rangle = \sigma|\psi_{k,\sigma}\rangle\,. \tag{8}$$

   Justify the following equality:

$$T|\psi_{k,\sigma}\rangle = \zeta_\sigma|\psi_{k,-\sigma}\rangle \tag{9}$$

   for some phase factor ζ_σ. What assumption did you need to make?

4. Recall that the raising and lowering operators in the spin j representation of $SU(2)$ act as

$$(J_1 \pm iJ_2)|\sigma\rangle = \sqrt{(j \mp \sigma)(j \pm \sigma + 1)}\,|\sigma \pm 1\rangle\,. \tag{10}$$

   Use this relation to derive a relation between the phases ζ_σ at different σ. Propose a simple solution for the relation you obtain.

5. Given the action of T on $|\psi_{k,\sigma}\rangle$, work out its action on $\zeta|\psi_{k,\sigma}\rangle$, for an arbitrary phase ζ. What can you conclude regarding the intrinsic time-reversal phase of massive particles?

6. For an arbitrary momentum p satisfying $p^2 = M^2$, use the explicit form of $L(p)$ given in the appendix to compute
$$\mathcal{T} L(p) \mathcal{T}^{-1} \,. \tag{11}$$
Use this to derive the action of T on $|\psi_{p,\sigma}\rangle$.

3 The axial anomaly

Consider the Lagrangian density of QED,
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi \,. \tag{12}$$

1. Work out the equations of motion for the fields $\bar{\psi}$ and ψ.

2. Write down the global $U(1)$ symmetry of $\mathcal{L}$ that underlies electromagnetism.

3. Recall that the field ψ transforms in the reducible Dirac representation of the Lorentz group. The operators $\frac{1}{2}(1 \pm \gamma^5)$ project onto the irreducible left- and right-moving Weyl representation,
$$\psi_L = \frac{1}{2}(1 - \gamma^5)\psi \,, \quad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi \,. \tag{13}$$
Rewrite $\mathcal{L}$ in terms of ψ_L and ψ_R.

4. Show that in the massless limit $m = 0$, the global $U(1)$ symmetry enhances to a $U(1) \times U(1)$ symmetry.

5. Show that the symmetry you identified in the previous question can be written as
$$\psi \to e^{i\alpha}\psi \,, \quad \psi \to e^{i\beta\gamma^5}\psi \,. \tag{14}$$

6. Derive the Noether currents (written in terms of the Dirac fields ψ and $\bar{\psi}$) corresponding to the two symmetries (14), and call them J^μ, J_5^μ respectively.

7. By invoking the equations of motion at $m \neq 0$, evaluate
$$\partial_\mu J^\mu \quad \text{and} \quad \partial_\mu J_5^\mu \,. \tag{15}$$
Does the result match your expectations?

It turns out that the symmetry giving rise to the current J_5^μ in the theory at $m = 0$ is anomalous. The rest of this problem is dedicated to demonstrating the violation of this symmetry quantum mechanically.

8. Consider the expression
$$iM_5^{\alpha\mu\nu}(p, q_1, q_2)(2\pi)^4\delta^4(p - q_1 - q_2) = \int d^4x\, d^4y\, d^4z\, e^{-ipx} e^{iq_1 y} e^{iq_2 z} \langle\Omega|T\{J_5^\alpha(x) J^\mu(y) J^\nu(z)\}|\Omega\rangle \,. \tag{16}$$
Write down the Feynman diagrams that contribute to $M_5^{\alpha\mu\nu}$ at lowest order.

9. To study the conservation properties of the current J_5^μ, we contract $M_5^{\alpha\mu\nu}$ by p_α (assuming $p \neq q_1, q_2$),

$$p_\alpha M_5^{\alpha\mu\nu}(p, q_1, q_2)(2\pi)^4\delta^4(p-q_1-q_2) = -\int d^4x\, d^4y\, d^4z\, e^{-ipx}e^{iq_1y}e^{iq_2z}\langle\Omega|T\{\partial_\alpha J_5^\alpha(x)J^\mu(y)J^\nu(z)\}|\Omega\rangle\,. \tag{17}$$

To leading order, this evaluates to

$$p_\alpha M_5^{\alpha\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\left[\frac{\mathrm{Tr}\left[\gamma^\mu \not{k}\gamma^\nu(\not{k}+\not{q}_2)\not{p}\gamma^5(\not{k}-\not{q}_1)\right]}{k^2(k+q_2)^2(k-q_1)^2} + (\mu \leftrightarrow \nu, 1 \leftrightarrow 2)\right]. \tag{18}$$

You are NOT asked to verify this result.

To simplify the above expression, first show that

$$\not{p}\gamma^5 = \gamma^5(\not{k}-\not{q}_1) + (\not{k}+\not{q}_2)\gamma^5\,. \tag{19}$$

10. Substitute (19) into (18) and show using symmetry arguments that if you close your eyes to convergence issues of the integrals, $p_\alpha M_5^{\alpha\mu\nu}$ vanishes.

11. To do better, we will use dimensional regularization. We need a prescription to deal with γ^5, which cannot be defined in arbitrary dimensions. We will follow 't Hooft and Veltman's prescription of maintaining the definition

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{20}$$

in arbitrary (fractional) dimension d, and formally decomposing a d dimensional momentum vector k as

$$k = k_4 + k_\epsilon\,, \tag{21}$$

such that

$$\begin{aligned} \not{k}\gamma^5 &= \left(\sum_{\mu=0}^{3}(k_4)_\mu\gamma^\mu + \sum_\eta (k_\epsilon)_\eta\gamma^\eta\right)\gamma^5 && (22)\\ &= \gamma^5\left(-\sum_{\mu=0}^{3}(k_4)_\mu\gamma^\mu + \sum_\eta (k_\epsilon)_\eta\gamma^\eta\right). && (23)\end{aligned}$$

The matrix γ^5 is hence taken to anti-commute with γ^μ for $\mu = 0, 1, 2, 3$, and to commute with the γ^η. The sum over the index η is formal: it extends over the fractional dimensions. We will extend the notation $\not{k}_\epsilon = \sum_\eta (k_\epsilon)_\eta\gamma^\eta$ to this case.

We will use the 't Hooft/Veltman prescription to evaluate (18). First, take the momentum k (which will be integrated over) to be in general dimension d, and all other occurring momenta to be purely four dimensional, and show that the RHS of equation (19), when expressed in terms of k and k_ϵ, picks up a correction term.

12. Substituting the result you just obtained in (18), show that this expression, with the k integration continued to d dimensions, evaluates to

$$p_\alpha M_5^{\alpha\mu\nu} = -16i\,\epsilon^{\mu\nu\alpha\beta}(q_1)_\alpha(q_2)_\beta \int \frac{d^dk}{(2\pi)^d}\frac{k_\epsilon^2}{k^2(k+q_2)^2(k-q_1)^2}\,. \tag{24}$$

You can use

$$(\not{k}_\epsilon)^2 = k_\epsilon^2 \tag{25}$$

in intermediate steps.

13. Simplify (24) using Feynman parameters.
14. Set
$$k_\epsilon^2 = \frac{d-4}{d} k^2 \tag{26}$$
under the $d^d k$ integral. Evaluate $p_\alpha M_5^{\alpha\mu\nu}$ in dimensional regularization in $d = 4 - \epsilon$ dimensions in the limit $\epsilon \to 0$.

A Useful formulae

The fiducial Lorentz transformation for $M > 0$

For $M > 0$, the fiducial Lorentz transformation $L(p)$ which maps $k = (M, 0, 0, 0)$ to p,
$$L(p)^\mu{}_\nu k^\nu = p^\mu \,, \tag{27}$$
can be chosen to be
$$L(p) = \begin{pmatrix} \gamma & \sqrt{\gamma^2-1}\hat{p}_i \\ \sqrt{\gamma^2-1}\hat{p}_j & \delta_{ij} + (\gamma-1)\hat{p}_i\hat{p}_j \end{pmatrix}_{i,j=1,2,3} \tag{28}$$
with $\gamma = \frac{\sqrt{\mathbf{p}^2+M^2}}{M}$ and $\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}$.

The fine structure constant

$$\alpha = \frac{e^2}{4\pi} \,. \tag{29}$$

Electron and positron polarization tensor

$$(\not{p} - m)u(p) = 0 \,, \quad (\not{p} + m)v(p) = 0 \,. \tag{30}$$

Weyl representation of γ matrices

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \tag{31}$$
where
$$\sigma^\mu = (1, \vec{\sigma}) \,, \quad \bar{\sigma}^\mu = (1, -\vec{\sigma}) \,, \tag{32}$$
and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices, which satisfy
$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \,. \tag{33}$$

Feynman parameters

$$\frac{1}{AB} = \int_0^1 dx\, dy\, \delta(x+y-1) \frac{1}{[xA+yB]^2} \,. \tag{34}$$
$$\frac{1}{ABC} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2}{[xA+yB+zC]^3} \,. \tag{35}$$

Evaluating integrals via Wick rotation

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2-\Delta+i\epsilon)^4} = \frac{-i}{48\pi^2}\frac{1}{\Delta}\,. \tag{36}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-\Delta+i\epsilon)^r} = i\frac{(-1)^r}{(4\pi)^2}\frac{1}{(r-1)(r-2)}\frac{1}{\Delta^{r-2}}\,, \quad r>2\,. \tag{37}$$

Dimensional regularization

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^{2a}}{(k^2-\Delta)^b} = i(-1)^{a-b}\frac{1}{(4\pi)^{d/2}}\frac{1}{\Delta^{b-a-\frac{d}{2}}}\frac{\Gamma(a+\frac{d}{2})\Gamma(b-a-\frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})}\,. \tag{38}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\,. \tag{39}$$

γ-matrices: properties and trace identities

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\,. \tag{40}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \quad (\gamma^5)^2 = 1\,. \tag{41}$$

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,. \tag{42}$$

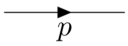
$$\mathrm{Tr}\,\gamma^5 = \mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}[\text{odd number of } \gamma\text{-matrices}] = 0\,. \tag{43}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}\,, \tag{44}$$

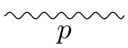
$$\mathrm{Tr}(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu) = 4\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right)\,, \tag{45}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^5) = -4i\,\epsilon^{\mu\nu\alpha\beta}\,. \tag{46}$$

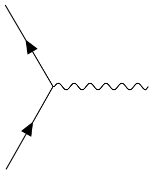
B Feynman rules for renormalized perturbation theory of QED



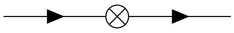
$$= \frac{i(\not{p} + m_R)}{p^2 - m_R^2 + i\epsilon}$$



$$= \frac{-i}{p^2 + i\epsilon}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

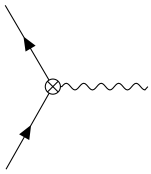


$$= -ie_R\gamma^\mu$$

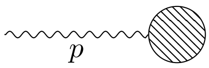


$$= i(\not{p}\delta_2 - (\delta_m + \delta_2)m_R)$$

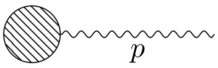
$$= -i\delta_3 p^2 g^{\mu\nu} \qquad \text{(in Feynman gauge)}$$



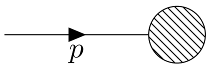
$$= -ie_R\delta_1\gamma^\mu$$



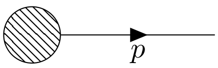
$= \epsilon_\mu(p)$ (incoming)



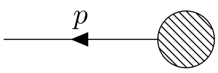
$= \epsilon_\mu^*(p)$ (outgoing)



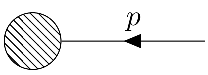
$= u^s(p)$ (incoming)



$= \bar{u}^s(p)$ (outgoing)



$= \bar{v}^s(p)$ (incoming)



$= v^s(p)$ (outgoing)