**ICFP Masters program**
École Normale Supérieure

**M1** *(Fall/Winter 2015/2016)*

## Final Exam: Mathematical Aspects of Symmetries in Physics

*This exam is closed book, closed notes. It consists of three pages. You have 3 hours. Carefully justify each step in your reasoning. Good luck!*

# 1 Finite groups and representations

1. **Conjugacy classes**

   Let $G$ denote a finite group. Recall that $x^G$ denotes the equivalence class of $x \in G$ with regard to conjugation, and $Z(G)$ denotes the center of $G$.

   (a) Prove that $|x^G| = 1 \Leftrightarrow x \in Z(G)$.

   (b) Prove the following equation, known as the *class equation*: Let $x_1, \ldots, x_l$ be representatives of the conjugacy classes of $G$. Then
   $$|G| = |Z(G)| + \sum_{x_i \notin Z(G)} |x_i^G| \,.$$

   (c) Let $p$ be a prime number (in French: *nombre premier*), and let $n$ be a positive integer. Suppose that $G$ is a group of order $p^n$.

   i. Use the class equation to show that $Z(G) \neq \{e\}$.

   ii. Suppose that $n \geq 3$ and that $|Z(G)| = p$. Prove that $G$ has a conjugacy class of size $p$.

2. **Equivariance, one dimensional representations**

   (a) Let $(\rho_V, V)$ and $(\rho_W, W)$ be two representations of a finite group $G$, and $\varphi : V \to W$ an invertible equivariant map. Let $\{v_i\}$ and $\{w_j\}$ be sets of basis vectors of $V$, $W$ respectively. Let $\mathcal{M}^V, \mathcal{M}^W, \mathcal{M}^\varphi$ be the matrix representations of the functions $\rho_V$, $\rho_W$, $\varphi$ with regard to the respective bases. Express $\mathcal{M}^W$ in terms of $\mathcal{M}^V$ and $\mathcal{M}^\varphi$.

   (b) Let $\rho_1, \rho_2 : G \to \mathbb{C} - \{0\}$ be one-dimensional representations of a finite group $G$. Show that $\rho_1$ is equivalent to $\rho_2$ if and only if $\rho_1 = \rho_2$.

   (c) Let $G$ be a finite group.

   i. Let $g \in G$. Prove that the set $\{g^1, g^2, \ldots, g^n\}$ for an appropriate positive integer $n$ carries a group structure. Characterize the minimal such $n$. It is called the order of the element $g$.

   ii. Let $\rho$ be a one-dimensional representation of the group $G$, and suppose that $g \in G$ has order $n$. Show that $\rho(g)$ is an $n^{th}$-root of unity.

   iii. Construct $n$ inequivalent one-dimensional representations of the group $\mathbb{Z}/n\mathbb{Z}$.

   iv. Explain why no further one-dimensional representations of $\mathbb{Z}/n\mathbb{Z}$ exist.

3. **Characters**

(a) A certain group $G$ of order 12 is known to have a total of four conjugacy classes, with representatives $g_1, \dots, g_4$. Three of its four irreducible characters take the following values:

| $g_i$ | $g_1 = e$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $\lvert C_G(g_i)\rvert$ | 12 | 4 | 3 | 3 |
| $\chi_1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | 1 | $\omega^2$ | $\omega$ |

We have here denoted $\omega = e^{2\pi i/3}$. Note that beneath each representative $g_i$ of a conjugacy class, we have given the order of the corresponding centralizer. Using the appropriate orthogonality relation of characters, find the complete character table of $G$. Justify each entry for $\chi_4$.

(b) Let $\psi$ be the class function on $G$ taking the following values:

| $g_i$ | $g_1 = e$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $\psi$ | 0 | 0 | $1 + 2\omega - \omega^2$ | $1 - \omega + 2\omega^2$ |

Express $\psi$ as a linear combination of $\chi_1, \dots, \chi_4$. Is $\psi$ a character of $G$?

# 2 Differential manifolds, Lie groups, and Lie algebras

1. **Differentiable structure on the real line**

Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas containing the chart $(\mathbb{R}, \varphi = id : \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas containing the chart $(\mathbb{R}', \psi : \mathbb{R}' \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

(b) Show that there exists a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. The two differentiable structures are thus equivalent.

2. **Vector spaces as differentiable manifolds**

Let $V$ be a $d$-dimensional real vector space. Let $\{e_i\}$ be a set of basis vectors for $V$, with dual basis $\{u_i\}$. Let $p \in V$, and $X \in T_pV$.

(a) Explain why the dual basis defines global coordinate functions on $V$.

(b) Write down an isomorphism between $T_pV$ and $V$.

(c) Assume $X$ has the expansion

$$X = \sum_{i=1}^{d} a_i \frac{\partial}{\partial u_i}|_p$$

in terms of the basis of $T_pV$ induced by the coordinate functions $u_i$. Let $\{\tilde{e}_i\}$ be a second set of basis vectors for $V$, with $e_i = \sum_{j=1}^{d} A_{ij}\tilde{e}_j$, $i = 1, \dots, d$. By acting on a $C^\infty$ function $f \in \mathcal{F}_p$, find the expression for $X$ in the basis induced by the dual basis $\{\tilde{u}_i\}$ to $\{\tilde{e}_i\}$.

(d) By considering the two expressions for $X$ thus obtained, prove that the isomorphism you introduced above between $T_pV$ and $V$ is basis independent.

3. **Lie algebra structure on $\mathbb{R}^2$**

   (a) Let $(x, y), (v, w) \in \mathbb{R}^2$. Define the bracket

   $$[(x, y), (v, w)] = (0, xw - yv)\,.$$

   Show that $(\mathbb{R}^2, [\cdot, \cdot])$ defines a Lie algebra.

   (b) Let $\mathfrak{g}$ be a 2-dimensional Lie algebra, spanned by basis vectors $\{e, f\}$ satisfying $[e, f] = f$. Prove that it is isomorphic as a Lie algebra to the one above.

   (c) Prove that up to isomorphism, only two different Lie algebra structures can be imposed on the vector space $\mathbb{R}^2$.

4. **Homomorphisms between Lie groups induce homomorphisms between Lie algebras**

   Let $\varphi : G \to H$ be a Lie group homomorphism. Then

   $$d\varphi : T_{e_G}G \to T_{e_H}H\,,$$

   as $\varphi(e_G) = e_H$. By means of the natural identifications $\mathfrak{g} \cong T_{e_G}G$, $\mathfrak{h} \cong T_{e_H}H$, $d\varphi$ thus induces a linear transformation of $\mathfrak{g}$ into $\mathfrak{h}$, which we also call $d\varphi$:

   $$d\varphi : \mathfrak{g} \to \mathfrak{h}\,.$$

   (a) Explain why for a given $X \in \mathfrak{g}$, $d\varphi(X)$ thus defined is the unique left-invariant vector field on $H$ such that

   $$d\varphi(X)(e_H) = d\varphi(X(e_G))\,.$$

   (b) Let $G$ and $H$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and let $\varphi : G \to H$ be a Lie group homomorphism. Prove the following:

      i. $X$ and $d\varphi(X)$ are $\varphi$-related for any $X \in \mathfrak{g}$.

      ii. The mapping $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.