M1- ICFP

Evolution and measurements of quantum states

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chapter 2: Basics Quantum Mechanics A reminder

1 state vectors

1.1 Postulates S1 and S2

let us consider a physical system
which is either isolated or submitted to fixed forces

– **(S1) a physical system, whatever its complexity, is described by a single mathematical object, called a *state vector*, or "ket", $|\Psi\rangle$, belonging to a well defined Hilbert space $\mathcal{H}$.**

– **(S2) the state vector, or "ket", is normalized to 1 :**

$$|\langle\Psi|\Psi\rangle|^2 = 1 \tag{1}$$

1.2 Properties

Let us consider an orthonormal basis $|u_n\rangle$ of space $\mathcal{H}$:

$$|\Psi\rangle = \sum_{n\in S} c_n |u_n\rangle \tag{2}$$

where the sum spans over a finite or infinite set S of integers. One has also :

$$\langle u_n|u'_n\rangle = \delta_{n,n'} \quad ; \quad \sum_{n\in S} |u_n\rangle\langle u_n| = \hat{1} \quad ; \quad c_n = \langle u_n|\Psi\rangle \tag{3}$$

In other instances, the Hilbert space has a continuous dimension, and one writes :

$$|\Psi\rangle = \int_I dx\, c(x)|u(x)\rangle \tag{4}$$

where the integral spans over a finite or infinite interval I of real numbers, with :

$$\langle u(x)|u(x')\rangle = \delta(x-x') \quad ; \quad \int_I dx\, |u(x)\rangle\langle u(x)| = \hat{1} \quad ; \quad c(x) = \langle u(x)|\Psi\rangle \tag{5}$$

Note that $|u(x)\rangle$ has not the same dimensionality as $|\Psi\rangle$ and does not belong to the same space.

1.3 The superposition "principle"

It follows from postulate S1 and the linear structure of the Hilbert space :

If $|\Psi_1\rangle$ *and* $|\Psi_2\rangle$ *describe two possible states of the physical system, then* $|\Psi_3\rangle = \lambda_1|\Psi_1\rangle + \lambda_2|\Psi_2\rangle$, *where* λ_1 *and* λ_2 *are any complex numbers (with* $|\lambda_1|^2 + |\lambda_2|^2 = 1$*) is another possible state of the physical system.*

The superposition principle implies the possibility of *quantum interferences* :

Example 1 : double slit experiment for photons or massive particles. If $|\Psi_1\rangle$ describes an electron passing through slit 1, and $|\Psi_2\rangle$ describes an electron passing through slit 2, then $|\Psi_3\rangle = (|\Psi_1\rangle + |\Psi_2\rangle)/\sqrt{2}$ describes another possible state of the electron, namely the one in which the electron has the possibility of passing through both slits.

Example 2 : an optical system which has a state $|\Psi_1\rangle$ which transmits light and a state $|\Psi_2\rangle$ which reflects it. The superposition $(|\Psi_1\rangle + |\Psi_2\rangle)/\sqrt{2}$ describes something which reflects and transmits light at the same time : in quantum optics and in contrast with literature, a door can be at the same time open **and** shut !

An extreme case is provided by the famous "Schrödinger cat", in which the two states (cat alive, cat dead) are distinct states of a macroscopic object.

1.4 Examples

Let us briefly mention here different systems that will be used as examples in the following :

- **Atoms** : They have an infinite number of well defined energy eigenstates $|\phi_n\rangle$ of energy E_n.
- **Qubits** : They are two level quantum systems, and are the quantum analog of the classical bits which can be in two macroscopically different states. Example : a spin 1/2 particle, which has two Zeeman substates that we will note $|+\rangle$ and $|-\rangle$. It is called a "qubit", which can be in the most general state is $|\Psi\rangle = \lambda_1|+\rangle + \lambda_2|-\rangle$. It is the basic ingredient of quantum information, a part of quantum physics which is the object of numerous investigations over the world because of its fundamental implications, but also because of possible applications in quantum computing.

- **1D particle** :
  Example : a massive particle which can travel along the x axis only. The ket $|x\rangle$ describes this particle when it is perfectly localized at point x. The general quantum state of such a particle is therefore the ket $|\Psi\rangle = \int dx\, \psi(x)|x\rangle$, where the complex function $\psi(x)$ is the wave function of the particle.
- **Light** :
  Example : a light beam of given propagation direction, polarization, frequency and transverse shape that is enclosed in a cavity. The Hilbert space has a basis of "number states" $|n\rangle$ describing a state of light containing exactly n photons inside the cavity. In particular the number state $|0\rangle$, called "vacuum" describes the obscurity.
  The most general state of this light beam is $|\Psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$. One can for example consider the quantum state $|0\rangle + |1\rangle)/\sqrt{2}$, a superposition of a single photon state and of vacuum.

2 Observables

2.1 Definition

To any real physical quantity A, like the energy, the position, the electric field, the magnetic moment, ... is associated a hermitian operator $\hat{A}$ operating in the Hilbert state of state vectors $\mathcal{H}$.

2.2 Properties

By definition $\hat{A}^\dagger = \hat{A}$, and $\hat{A}$ can be diagonalized :

$$\hat{A}|u_n\rangle = a_n|u_n\rangle \tag{6}$$

the eigenvalues a_n being real, and the eigenvectors $|u_n\rangle$ forming an orthonormal basis of the Hilbert space $\mathcal{H}$.

In some instances, in the so-called degenerate cases, several orthogonal eigenvectors $|u_n^i\rangle$ are associated with the same eigenvalue a_n. One the defines textitprojectors on eigenspaces $\hat{P}_n$ by :

$$\hat{P}_n = \sum_i |u_n^i\rangle\langle u_n^i| \tag{7}$$

They are such that

$$\hat{P}_n\hat{P}_{n'} = \delta_{n,n'}\hat{P}_n \quad ; \quad \sum_n \hat{P}_n = \hat{1} \quad ; \quad \sum_n a_n\hat{P}_n = \hat{A} \tag{8}$$

2.3 Examples

- **atoms** :
  With the atom energy E is associated the hermitian operator hamiltonian $\hat{H}$ such that :
  $$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle \tag{9}$$

- **qubits** :
  With the three components of the spin angular momentum $\mathbf{S}$ of a spin 1/2 particle are associated three hermitian operators $\hat{S}_i$ $(i = x, y, z)$ which are respectively equal to $\sigma_i/2$ where σ_i $(i = x, y, z)$ are the Pauli matrices :
  $$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$
  the first and second columns corresponding respectively to qubit basis vectors $|+\rangle$ and $|-\rangle$. These matrices have the following properties :
  $$\sigma_i^2 = \hat{1} \quad ; \quad \sigma_i\sigma_j = -\sigma_j\sigma_i = i\sigma_k \tag{11}$$
  when (i, j, k) form a direct permutation of (x, y, z).
  The following non-hermitian operators, called spin flip operators, are also useful :
  $$\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |+\rangle\langle-| \quad ; \quad \sigma_- = \sigma_+^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |-\rangle\langle+| \tag{12}$$

– **1D particle** :
With the position x of the particle is associated the hermitian operator $\hat{X}$ such that :

$$\hat{X}|x\rangle = x|x\rangle \tag{13}$$

With the momentum p of the 1D particle is associated the hermitian operator $\hat{P}$ such that :

$$\hat{P}|p\rangle = p|p\rangle \tag{14}$$

On can show that :

$$[\hat{X}, \hat{P}] = ih\hat{1} \quad ; \quad \langle x|p\rangle = \frac{1}{2\pi\hbar} e^{ipx/\hbar} \tag{15}$$

– **light** : With the photon number N is associated the number operator $\hat{N}$ such that :

$$\hat{N}|n\rangle = n|n\rangle \tag{16}$$

the electromagnetic energy being associated with the hamiltonian operator $\hat{H} = (\hat{N} + 1/2)\hbar\omega$.
In analogy with the harmonic oscillator one introduces the non hermitian photon annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ such that

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle; \quad \hat{a}|0\rangle = 0; \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle; \quad \hat{N} = \hat{a}^\dagger\hat{a} \tag{17}$$

With the electric field $\mathbf{E}(\mathbf{r}, \mathbf{t})$ is associated the hermitian operator $\hat{\mathbf{E}}(\mathbf{r}, \mathbf{t})$ which can be written as :

$$\hat{\mathbf{E}}(\mathbf{r}, \mathbf{t}) = \hat{\mathbf{E}}^{+}(\mathbf{r}, \mathbf{t}) + (\hat{\mathbf{E}}^{+}(\mathbf{r}, \mathbf{t}))^\dagger \tag{18}$$

with

$$\hat{\mathbf{E}}^{+}(\mathbf{r}, \mathbf{t}) = \mathcal{E}\hat{\mathbf{a}}\,\varepsilon\,\mathbf{e}^{\mathbf{i}(\mathbf{k}\cdot\mathbf{r}-\omega\mathbf{t})} \tag{19}$$

$\mathcal{E}$ being the electric field of a single photon and ε the polarization unit vector of the field.
In short, the wave properties of light (electric field) depend linearly on $\hat{a}$ and $\hat{a}^\dagger$, whereas its particle properties (photon number) depend on $\hat{a}^\dagger\hat{a}$.

3 Measurements

3.1 Postulates M1, M2 and M3

They concern ideal measurements, that we will call Von Neumann, or projective, measurements :

- **(M1) The measurement of the physical quantity A can only give as a result one of the eigenvalues a_n of the associated observable $\hat{A}$.**

- **(M2) In general the precise result of the measurement cannot be predicted with certainty. The probability $Proba(a_n||\psi\rangle)$ of obtaining the result a_n when the system is in the quantum state described by the state vector $|\psi\rangle$ is given by Born's rule :**

$$Proba(a_n||\psi\rangle) = \langle\psi|\hat{P}_n|\psi\rangle \tag{20}$$

where $\hat{P}_n$ is the projector on the eigen subspace associated with eigenvalue a_n.

– **(M3) The conditional state of the system just after the measurement and *and in the subset of measurements that have given the result* a_n is :**

$$|\psi^{after|a_n}\rangle\rangle = \frac{\hat{P}_n|\psi\rangle}{\|\hat{P}_n|\psi\rangle\|} \tag{21}$$

This state is called a *conditional state* because it is not the one which is always obtained after the measurement, but only in the special and uncontrolled opportunities when the measurement gives the precise result a_n

We will call the three postulates M1, M2, M3 respectively the *quantization postulate, the Born rule postulate, and the state collapse postulate.*

3.2 Consequences of M1 and M2

Let us now assume that the preparation process is such that it delivers many identical copies of the system in state $|\psi\rangle$. One can then make many measurements of the same physical quantity A, and the probability tends then to certainty if one is interested only in the statistical moments of the probability distribution of the measured values. In particular the mean value of A, that we will note $< A >$, is :

$$< A >= \sum_n a_n Proba(a_n||\psi\rangle) = \langle\psi|\hat{A}|\psi\rangle \tag{22}$$

as well as the mean of any function $f(A)$ of A :

$$< f(A) >= \langle\psi|f(\hat{A})|\psi\rangle \tag{23}$$

3.3 Consequences of M3

Case of two successive measurements of quantity A on the same system

(not to be confused with two measurements of A on identically prepared quantum states).

- the probability of getting for the second measurement a result $a_{n'}$ different from the first result a_n is zero

- the probability of getting for the second measurement the same result a_n as the first result a_n is one.

Probabilities are in this special case certainties : after the first measurement we know with certainty the result of the measurement of A on the system. The first measurement acts as a *state preparation* : quantum physics has some intrinsic randomness, but is at least repeatable !

The state after the measurement is very different from the state before it. The measurement process results in a *very strong perturbation of the system*, during which most of the information about the initial state is lost. This is called the '**"state collapse"**

Consequence : it is not possible to perfectly determine from successive measurements the quantum state of a system when its preparation is not known.

4 Description of composite systems

4.1 Tensor product of Hilbert spaces

Particle 1 is described by a state vector $|\psi_1\rangle$ belonging to a Hilbert space $\mathcal{H}_1$ of dimension d_1 spanned by a basis of vectors $|u_i\rangle$ $(i = 1, ..., d_1)$

Particle 2 is described by a state vector $|\psi_2\rangle$ belonging to a Hilbert space $\mathcal{H}_2$ of dimension d_2 spanned by a basis of vectors $|v_j\rangle$ $(j = 1, ..., d_2)$

The system formed by the two particles is described by a state vector $|\Psi\rangle$ belonging to the *tensor product* $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

This space has a dimension $d_1 \times d_2$ (and not $d_1 + d_2$), and is spanned by the basis of vectors $|u_i\rangle \otimes |v_j\rangle$ $(i = 1, ..., d_1, j = 1, ..., d_2)$, that we will write $|u_i, v_j\rangle$ for simplicity. It has a Hilbert inner product defined on its vector basis by :

$$\langle u_i, v_j | u_{i'}, v_{j'}\rangle = \delta_{i,i'}\delta_{j,j'} \tag{24}$$

A state belonging to $\mathcal{H}$ has therefore the form

$$|\Psi\rangle = \sum_{i,j} \lambda_{i,j} |u_i, v_j\rangle \tag{25}$$

Note that it cannot be always written in a factorized way, as

$$|\Psi\rangle = \sum_{i,j} \lambda_i^1 \lambda_j^2 |u_i, v_j\rangle = \left(\sum_i \lambda_i^1 |u_i\rangle\right) \otimes \left(\sum_j \lambda_j^2 |v_j\rangle\right) \tag{26}$$

If such a factorization is not possible, the state is said **entangled**, or *intriqué* in french.

In the case of two 1D particles considered in the previous section, a factorized state has a wave function of the form $\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$, whereas an entangled state is characterized by a wave function which cannot be written as a product.

4.2 Operators in tensor product of Hilbert spaces

Let us consider an operator $\hat{A}_1$ operating on Hilbert space $\mathcal{H}_1$, and an operator $\hat{B}_2$ operating on Hilbert space $\mathcal{H}_2$. The operator $\hat{A}_1 \otimes \hat{B}_2$ operating on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is defined by :

$$\hat{A}_1 \otimes \hat{B}_2 |u_i, v_j\rangle = \left(\hat{A}_1 |u_i\rangle\right) \otimes \left(\hat{B}_2 |v_j\rangle\right) \tag{27}$$

In particular, one can extend an operator $\hat{A}_1$ acting on space $\mathcal{H}_1$ to an operator acting on tensor space $\mathcal{H}$ by :

$$\hat{A}_1 |u_i, v_j\rangle = \left(\hat{A}_1 |u_i\rangle\right) \otimes |v_j\rangle \tag{28}$$

Note that one always has for the commutators between operators acting on different subspaces :

$$\left[\hat{A}_1, \hat{B}_2\right] = 0 \quad \forall \hat{A}_1 \, \forall \hat{B}_2 \tag{29}$$

For example in the case of two 1D particles $[\hat{x}_1, \hat{p}_2] = 0$ whereas $[\hat{x}_1, \hat{p}_1] = i\hbar$.

4.3 Example

The system formed by two qubits, labeled 1 and 2, which is of dimension 4.

A possible basis of the corresponding Hilbert space is the set $\{|1+, 2+\rangle, |1-, 2+\rangle, |1-, 2+\rangle, |1-, 2-\rangle,$

Another useful basis is the set of entangled so-called "Bell states" $\{|\Psi_{\pm}\rangle, |\Phi_{\pm}\rangle\}$ defined by

$$|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|1+, 2-\rangle \pm |1-, 2+\rangle) \tag{30}$$

$$|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|1+, 2+\rangle \pm |1-, 2-\rangle) \tag{31}$$