Problem Set for Exercise Session No.4

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

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1 Character for Representation of D_3

(1) Let us consider a finite group G of order r and its d-dimensional representation ρ. Since ρ is completely reducible, one can decompose ρ's into a direct sum of irreducible representations $\rho^{(a)}$: $\rho = \oplus_a m_a \rho^{(a)}$. Here m_a counts how many times the representation $\rho^{(a)}$ appears in ρ. We denote representation matrices of $\rho(g)$ and $\rho^{(a)}(g)$ as $M(g)$ and $M^{(a)}(g)$, respectively ($g \in G$). Then the characters corresponding to them are defined by $\chi^{(\rho)}(g) = \mathrm{tr}(M(g))$ and $\chi^{(a)}(g) = \mathrm{tr}(M^{(a)}(g))$ respectively.

Prove the following relation:

$$m_a = \frac{1}{r}\sum_{i=1}^{n_c} r_i \chi(C_i)\,\overline{\chi^{(a)}(C_i)}\,.$$

Here C_i is a conjugacy class of G with a representative $g_i \in G$ $(i = 1, 2, \cdots, n_c)$, n_c is the number of the conjugacy classes and r_i is the number of elements in C_i. We notice that the characters satisfy $\chi(g) = \chi(g_i)$ and $\chi^{(a)}(g) = \chi^{(a)}(g_i)$ for $g \in C_i$. Thus we denoted the characters corresponding to the representations ρ and $\rho^{(a)}$ for an element in C_i as $\chi(C_i)$ and $\chi^{(a)}(C_i)$, respectively.

(2) Let us consider the characters for the representations of the group D_3. As we have shown in Problem 2 of Problem Set No.1, there are three conjugacy classes of D_3:

$$C_1 = \{e\}\,, \qquad C_2 = \{c_3, c_3^{-1}\}\,, \qquad C_3 = \{\sigma_1, \sigma_2, \sigma_3\}\,.$$

Answer the following questions :

1. For the three-dimensional representation ρ given in Problem 1 of Problem Set No.2, calculate the characters.

2. For the irreducible representations $\rho^{(1)}$, $\rho^{(1')}$ and $\rho^{(2)}$ of D_3, calculate the characters. Confirm the orthogonality of the characters for the irreducible representations explicitly.

3. Explain that the irreducible representations of D_3 are $\rho^{(1)}$, $\rho^{(1')}$ and $\rho^{(2)}$ only (up to those isomorphic to these three).

4. Calculate m_1, $m_{1'}$ and m_2 for the representation ρ.

2 Induced Representation

Let us consider a finite group G and its subgroup H. One can carry out the right coset decomposition of G as

$$G = g_1 H + g_2 H + \cdots + g_p H\,,$$

where $g_1, g_2, \cdots, g_p \in G$ and $g_i H \cap g_j H = \emptyset$ for $i \neq j$. Let us next take a n_H-dimensional representation of H and denote its matrix representation as $m(h) = (m_{ab}(h))$ for $h \in H$ $(a, b = 1, 2, \cdots n_H)$. From this representation of H, one can construct a pn_H-dimensional representation of G in the following way: for $g \in G$, $(n_H(i-1)+a, n_H(j-1)+b)$-component of the matrix representation $M(g)$ is defined as (we denote it as $M_{ia,jb}(g)$ here)

$$M_{ia,jb}(g) = \begin{cases} m_{ab}(g_i^{-1} g g_j) & \text{for} \quad g_i^{-1} g g_j \in H\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We note that $i, j = 1, 2, \cdots, p$ (and thus $M(g)$ is a $pn_H \times pn_H$ matrix). This representation of G is called the induced representation.

(1) Show that the induced representation is indeed a representation of G (that is, confirm that $M(g)M(g') = M(gg')$ for $g, g' \in G$).

Now we consider irreducible representations $\rho^{(\alpha)}$ of H and denote its matrix representation as $m^{(\alpha)}(h)$ for $h \in H$. Then from these, one can construct the induced representations for G as above. We denote the corresponding matrix representations as $M^{(\alpha)}_{ind}(g)$ (for $g \in G$).

Let us next introduce the irreducible representations of G denoted by $\rho_G^{(A)}$ for which the matrix representation is given by $M_G^{(A)}(g)$ (for $g \in G$). We can decompose the character for the induced representation by using the ones for the irreducible representation as

$$\chi^{(\alpha)}_{ind}(g) = \sum_{A:\text{irrep. of } G} n_A^{(\alpha)} \chi_G^{(A)}(g)\,,$$

where $\chi^{(\alpha)}_{ind}(g)$ and $\chi_G^{(A)}(g)$ are the characters corresponding to $M^{(\alpha)}_{ind}(g)$ and $M_G^{(A)}(g)$, respectively, and $n_A^{(\alpha)}$ counts how many time each irreducible representation of G appears in the induced representation.

On the other hand, we can construct a representation of H by restricting $\rho_G^{(A)}$ to H. Then the decomposition of the corresponding character into the ones for the irreducible representation of H is written as (for $h \in H$)

$$\chi_G^{(A)}(h) = \sum_{\alpha:\text{irrep. of } H} n_\alpha^{(A)} \chi^{(\alpha)}(h)\,,$$

Here $\chi^{(\alpha)}(h)$ is the character corresponding to $m^{(\alpha)}(h)$ and $n_\alpha^{(A)}$ counts how many times each irreducible representation of H appears in this representation constructed by restricting $\rho_G^{(A)}$ to H.

(2) Prove the Frobenius reciprocity relation for these two decomposition coefficients:

$$n_A^{(\alpha)} = n_\alpha^{(A)} .$$