Solution Set for Exercise Session No.2

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

27th November, 2014, at Room 235A

Lecture by Amir-Kian Kashani-Poor (email: kashani@lpt.ens.fr)
Exercise Session by Tatsuo Azeyanagi (email: tatsuo.azeyanagi@phys.ens.fr)

1 Representation of D_3

1. Let us check two cases explicitly:

$$c_3\,\mathbf{e}_1 = \mathbf{e}_2\,, \qquad c_3\,\mathbf{e}_2 = \mathbf{e}_3\,, \qquad c_3\,\mathbf{e}_3 = \mathbf{e}_1\,,$$
$$\sigma_1\,\mathbf{e}_1 = \mathbf{e}_1\,, \qquad \sigma_1\,\mathbf{e}_2 = \mathbf{e}_3\,, \qquad \sigma_1\,\mathbf{e}_3 = \mathbf{e}_2\,,$$

which lead to (from the definition of $R_{ij}(g)$)

$$R_{21}(c_3) = 1\,, \qquad R_{32}(c_3) = 1\,, \qquad R_{13}(c_3) = 1\,,$$
$$R_{11}(\sigma_1) = 1\,, \qquad R_{32}(\sigma_1) = 1\,, \qquad R_{23}(\sigma_1) = 1\,,$$

while the other components of $R_{ij}(c_3)$ and $R_{ij}(\sigma_1)$ are zero. In the form of matrices, we thus obtain

$$R(c_3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\,, \qquad R(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,.$$

We can compute the other $R(g)$'s in the same way.

2. These two bases are related as

$$(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \equiv (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) N\,.$$

The inverse of N is computed as

$$N^{-1} = N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\,.$$

Since

$$g\,(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3) = (\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)\tilde{R}(g) \quad \leftrightarrow \quad g\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)N = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)N\tilde{R}(g)$$
$$\leftrightarrow \quad g\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)N\tilde{R}(g)N^T\,,$$

we obtain

$$R(g) = N\tilde{R}(g)N^T \qquad \leftrightarrow \qquad \tilde{R}(g) = N^T R(g) N\,.$$

Here we evaluate some $\tilde{R}(g)$'s explicitly:

$$\begin{aligned}
\tilde{R}(c_3) &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \\
\tilde{R}(\sigma_1) &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned}$$

The other $\tilde{R}(g)$'s are computed in a similar way.

3. $\rho^{(1)}$ is unitary representation since $(R^{(1)}(g))^{-1} = R^{(1)}(g) = (R^{(1)}(g))^{\dagger}$. $\rho^{(2)}$ is also unitary (one can check one by one).

4. We can multiply $R^{(1')}(g_1)$ and $R^{(1')}(g_2)$ ($g_1, g_2 \in D_3$) to get the following table.

	$R^{(1')}(e)$	$R^{(1')}(c_3)$	$R^{(1')}(c_3^{-1})$	$R^{(1')}(\sigma_1)$	$R^{(1')}(\sigma_2)$	$R^{(1')}(\sigma_3)$
$R^{(1')}(e)$	1	1	1	−1	−1	−1
$R^{(1')}(c_3)$	1	1	1	−1	−1	−1
$R^{(1')}(c_3^{-1})$	1	1	1	−1	−1	−1
$R^{(1')}(\sigma_1)$	−1	−1	−1	1	1	1
$R^{(1')}(\sigma_2)$	−1	−1	−1	1	1	1
$R^{(1')}(\sigma_3)$	−1	−1	−1	1	1	1

Table 1: Multiplication table for $R^{(1')}$

From this multiplication table derived in Problem Set No.1, we can see that this $\rho^{(1')}$ is a one-dimensional irreducible epresentation of D_3 (for example $R^{(1')}(c_3)R^{(1')}(\sigma_1) = -1 = R^{(1')}(\sigma_3) = R^{(1')}(c_3\sigma_1)$ etc).