Solution Set for Exercise Session No.6

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

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1 Differential

(1)

1. Let us take $v \in T_pM_1$ and C^∞ function h defined in the neighborhood of $(g \circ f)(p)$. Then we have

$$d(g \circ f)_p(v)h = v(h \circ g \circ f) = df_p(v)(h \circ g) = dg_{f(p)}(df_p(v))h = dg_{f(p)} \circ df_p(v)h\,.$$

2. Let us take $v \in T_pM_1$ and C^∞ function h defined in the neighborhood of $id_{M_1}(p) = p$. Then we have

$$d(id_{M_1})_p(v)h = v(h \circ id_{M_1}) = v(h) = id_{T_pM_1}(v)h\,.$$

   Here we have used $id_{T_pM_1}(v) = v$.

3. By using the result in the first and second problems we obtain

$$\begin{aligned} d(f^{-1})_{f(p)} \circ df_p &= d(f^{-1} \circ f)_p = d(id_{M_1})_p = id_{T_pM_1}\,, \\ df_p \circ d(f^{-1})_{f(p)} &= d(f \circ f^{-1})_{f(p)} = d(id_{M_2})_{f(p)} = id_{T_{f(p)}M_2}\,. \end{aligned}$$

(2) Let us recall the coordinate for $\mathbb{R}P^2$ introduced in Problem Set No.5. The atlas $\{(U_x, \phi_x), (U_y, \phi_y), (U_z, \phi_z)\}$ is defined with the open sets defined by

$$U_x = \{[x : y : z] \in \mathbb{R}P^2 | x \neq 0\}\,,\ \ U_y = \{[x : y : z] \in \mathbb{R}P^2 | y \neq 0\}\,,\ \ U_z = \{[x : y : z] \in \mathbb{R}P^2 | z \neq 0\}\,,$$

and the maps from these open sets to $\mathbb{R}^2$ defined by (for later convenience we denoted the coordinates as ϕ_x, ϕ_y, ϕ_z (while in Problem Set No.5 we denoted as $\varphi_x, \varphi_y, \varphi_z$))

$$\phi_x([x : y : z]) = \left(\frac{y}{x},\ \frac{z}{x}\right)\,, \qquad \phi_y([x : y : z]) = \left(\frac{x}{y},\ \frac{z}{y}\right)\,, \qquad \phi_z([x : y : z]) = \left(\frac{x}{z},\ \frac{y}{z}\right)\,.$$

We also note that the inverses of ϕ_x, ϕ_y, ϕ_z are

$$\begin{aligned} (\phi_x)^{-1}(Y, Z) &= [1 : Y : Z]\,, \qquad \text{for } (Y, Z) \in \mathbb{R}^2\,, \\ (\phi_y)^{-1}(X, Z) &= [X : 1 : Z]\,, \qquad \text{for } (X, Z) \in \mathbb{R}^2\,, \\ (\phi_z)^{-1}(X, Y) &= [X : Y : 1]\,, \qquad \text{for } (X, Y) \in \mathbb{R}^2\,. \end{aligned}$$

1. Let us first consider $(Y, Z) \in \phi_x(U_x)$. Then we have

$$\psi \circ \phi_x^{-1}(Y, Z) = \phi([1 : Y : Z]) = \frac{Z^2}{1 + Y^2 + Z^2} ,$$

and thus we have

$$\frac{\partial(\phi \circ \phi_x^{-1})}{\partial Y} = -\frac{2YZ^2}{(1 + Y^2 + Z^2)^2} ,$$
$$\frac{\partial(\phi \circ \phi_x^{-1})}{\partial Z} = \frac{2Z}{1 + Y^2 + Z^2} - \frac{2Z^3}{(1 + Y^2 + Z^2)^2} = \frac{2Z(1 + Y^2)}{(1 + Y^2 + Z^2)^2} .$$

Thus the differential vanishes at $[1 : Y : 0]$ for any Y. In a similar way, we can show that for $(X, Z) \in \phi_y(U_y)$, the differential vanishes at $[X : 1 : 0]$ for any X.

We next consider $(X, Y) \in \phi_z(U_z)$. In this case, we have

$$\psi \circ \phi_z^{-1}(X, Y) = \psi([X : Y : 1]) = \frac{1}{1 + X^2 + Y^2} ,$$

and then its derivatives are

$$\frac{\partial(\psi \circ \phi_z^{-1})}{\partial X} = -\frac{2X}{(1 + X^2 + Y^2)^2} ,$$
$$\frac{\partial(\psi \circ \phi_z^{-1})}{\partial Y} = -\frac{2Y}{(1 + X^2 + Y^2)^2} .$$

Thus the differential vanishes at $[0 : 0 : 1]$.

To summarize, we have found that the differential $d\psi_p$ vanishes at $p = [1 : Y : 0], [X : 1 : 0], [0 : 0 : 1]$ where $X, Y \in \mathbb{R}$.

2. For $S^2 = \{(x, y, z)|x^2 + y^2 + z^2 = 1\}$, we use the coordinates introduced in Problem 1-(1)-1, that is, we consider the atlas $\{(O_x^\pm, \varphi_x^\pm), (O_y^\pm, \varphi_y^\pm), (O_z^\pm, \varphi_z^\pm)\}$ where the open sets are defined by

$$O_x^+ = \{(x, y, z) \in S^2|x > 0\} , \qquad O_x^- = \{(x, y, z) \in S^2|x < 0\} ,$$
$$O_y^+ = \{(x, y, z) \in S^2|y > 0\} , \qquad O_y^- = \{(x, y, z) \in S^2|y < 0\} ,$$
$$O_z^+ = \{(x, y, z) \in S^2|z > 0\} , \qquad O_z^- = \{(x, y, z) \in S^2|z < 0\} .$$

and $\varphi_x^\pm, \varphi_y^\pm, \varphi_z^\pm$ are maps from the corresponding open sets to a two-dimensional open disk $D^2 = \{(a, b) \in \mathbb{R}^2|a^2 + b^2 < 1\}$:

$$\varphi_x^+(x, y, z) = \varphi_x^-(x, y, z) = (y, z) ,$$
$$\varphi_y^+(x, y, z) = \varphi_y^-(x, y, z) = (x, z) ,$$
$$\varphi_z^+(x, y, z) = \varphi_z^-(x, y, z) = (x, y) .$$

We also recall that the inverses of these maps are

$$(\varphi_x^\pm)^{-1}(a, b) = (\pm(1 - a^2 - b^2)^{1/2}, a, b) ,$$

$$(\varphi_y^{\pm})^{-1}(a,b) = (a, \pm(1-a^2-b^2)^{1/2}, b)\,,$$
$$(\varphi_z^{\pm})^{-1}(a,b) = (a, b, \pm(1-a^2-b^2)^{1/2})\,.$$

Here $(a,b) \in D^2 = \{(a,b) \in \mathbb{R}^2 | a^2+b^2 < 1\}$. For $\mathbb{R}P^2$, we introduce the coordinates in the same way as the previous problem.

Then we can have

$$\begin{aligned}
\phi_x \circ \pi \circ (\varphi_x^{\pm})^{-1}(a,b) &= \phi_x \circ \pi \left(\pm(1-a^2-b^2)^{1/2}, a, b\right) \\
&= \pm\left(\frac{a}{(1-a^2-b^2)^{1/2}}, \frac{b}{(1-a^2-b^2)^{1/2}}\right)\,, \\
\phi_y \circ \pi \circ (\varphi_y^{\pm})^{-1}(a,b) &= \phi_y \circ \pi \left(a, \pm(1-a^2-b^2)^{1/2}, b\right) \\
&= \pm\left(\frac{a}{(1-a^2-b^2)^{1/2}}, \frac{b}{(1-a^2-b^2)^{1/2}}\right)\,, \\
\phi_z \circ \pi \circ (\varphi_z^{\pm})^{-1}(a,b) &= \phi_z \circ \pi \left(a, b, \pm(1-a^2-b^2)^{1/2}\right) \\
&= \pm\left(\frac{a}{(1-a^2-b^2)^{1/2}}, \frac{b}{(1-a^2-b^2)^{1/2}}\right)\,.
\end{aligned}$$

Thus we get the Jacobian for $\phi_i \circ \pi \circ (\varphi_i^{\pm})$ (where $i = x, y, z$) as follows:

$$\pm\frac{1}{(1-a^2-b^2)^{3/2}}\begin{pmatrix} 1-b^2 & ab \\ ab & 1-a^2 \end{pmatrix}\,.$$

Then the determinant to this matrix is given by

$$\frac{(1-b^2)(1-a^2)-a^2b^2}{(1-a^2-b^2)^3} = \frac{1-a^2-b^2}{(1-a^2-b^2)^3} = \frac{1}{(1-a^2-b^2)^2}\,,$$

which is nonzero for any $(a,b) \in D^2$. We have confirmed that the Jacobian is not degenerate at any $(a,b) \in D^2$. At a point $p \in O_i^{\pm}$ ($i = x, y, z$), the Jacobian for $\phi_i \circ \pi \circ (\varphi_i^{\pm})^{-1}$ at $\varphi_i^{\pm}(p)$ gives a coordinate expression of the differential map $d\pi_p$, and thus the non-degenerate Jacobian means that the differential map $d\pi_p$ is isomorphic. We thus conclude that at any point $p \in S^2$, $d\pi_p$ is isomorphic.