Lecture II & III: Supplements

THE LANGEVIN EQUATION

A Brownian particle is subjected to an external force due to gravity (or to an electric field manipulating the particle), and to forces coming from the surrounding fluid, due to collisions with the molecules. Langevin's idea was to decompose these latter forces into an *average component* corresponding to friction, and a *fluctuating component* with very small correlation time, corresponding to the randomness of the momentum of impinging molecules. For simplicity, let us consider a Brownian particle confined to move in one dimension, X. The Newton equation of motion postulated by Langevin is:

$$m\frac{dV(t)}{dt} = -\gamma V(t) + F(X) + \eta(t), \qquad V(t) = \frac{dX(t)}{dt} = \dot{X}(t) \tag{1}$$

where m is the mass of the particle, γ is the friction coefficient, $F(X)$ the external force and $\eta(t)$ the Langevin noise defined as a Gaussian random function, of zero mean and correlator given by:

$$G(t,t') := \langle \eta(t)\eta(t')\rangle = \frac{\sigma_0^2}{2\tau_c}\exp\left(-\frac{|t-t'|}{\tau_c}\right). \tag{2}$$

with a very small *correlation time* $\tau_c \to 0$. Note that the *order of magnitude* of η is $\sigma_0/\sqrt{\tau_c}$.

In the following, we will consider that m is very small such that the inertial term can be neglected. In this so called "over-damped limit", the Langevin equation takes a simpler form:

$$\gamma\frac{dX(t)}{dt} = F(X) + \eta(t). \tag{3}$$

One can always choose the units of X such that $\gamma = 1$. The over-damped Langevin equation says that the velocity of the particle $\dot{X}$ is given by a systematic term $F(X)$ related to the external drive and a fluctuating term which is the Gaussian Langevin noise with the characteristics discussed above.

ITÔ VS. STRATONOVICH PRESCRIPTIONS

The precise meaning of the Langevin equation $\dot{X}(t) = F(X) + \eta(t)$ deserves special attention. We shall not go into all details of its various mathematical definitions, but it is important to have an idea of the source of ambiguity inherent in writing such an equation. Because the noise $\eta(t) \sim \tau_c^{-1/2}$ is divergent in the zero correlation time limit, the Langevin equation can be ambiguous, in particular in more general situations where η is multiplied by a function of X.

The physical origin of this ambiguity is that there are *two time scales* that go to zero in this problem. One is the infinitesimal (discretization) time scale $\mathrm{d}t$, the other one is the correlation time of the Langevin noise τ_c which, in real physical situations, can never be zero. The "physical" choice therefore corresponds to τ_c small but fixed, whereas $\mathrm{d}t \to 0$ is a mathematical convenience allowing one to write a differential equation of motion.

Stratonovich prescription: $\tau_c \gg \mathrm{d}t$. If τ_c does not go to zero, then η is a perfectly regular function of time. Therefore the standard rules of differential calculus apply. For example, for any function $f(X,t)$:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X}\dot{X}, \tag{4}$$

where $\dot{X}$ is given by the Langevin equation Eq. (3). But because τ_c is not zero, the noise at time t, $\eta(t)$, cannot be assumed to be uncorrelated from the past values $X(t')$, $t' \le t$. Therefore, averages like $\langle g(X(t))\eta(t)\rangle$ are not zero. They are actually given by the following important *Stratonovitch rule*:

$$\langle g(X(t))\eta(t)\rangle = \frac{\sigma_0^2}{2}\langle g'(X(t))\rangle, \qquad (\tau_c \to 0). \tag{5}$$

Let us sketch the proof of this identity. Because η is Gaussian, one can show that for any function $g(X)$:

$$\langle g(X(t))\eta(t)\rangle = \int_{-\infty}^{\infty} \mathrm{d}t' \left\langle \frac{\partial g(X(t))}{\partial \eta(t')} \right\rangle G(t,t'), \tag{6}$$

where $G(t,t')$ is given in Eq. (2). [Here we use a continuous-time notation, and the derivative $\partial g(X(t))/\partial \eta(t')$ is a functional derivative. This should not be particularly scary. It can be understood as the continuous time limit of a discretized process, where $t = n\tau$.]

If τ_c is very small, because of the structure (2) of the correlator $G(t,t')$, only t' close to t will matter in the integral. Now,

$$\frac{\partial g(X(t))}{\partial \eta(t')} = g'(X(t))\frac{\partial X(t)}{\partial \eta(t')}, \tag{7}$$

and using the short time expansion of the Langevin equation, $X(t) \approx X(t') + F(X(t))(t-t') + \int_{t'}^{t} \mathrm{d}t'' \eta(t'') + O((t-t')^2)$. This allows one to obtain, up to higher order corrections in $t - t'$:

$$\frac{\partial X(t)}{\partial \eta(t')} \approx 1 + F'(X(t))(t-t') + \ldots \qquad (t' < t), \tag{8}$$

and therefore, in the limit $\tau_c \to 0$:

$$\langle g(X(t))\eta(t)\rangle \approx \int_{-\infty}^{t} \mathrm{d}t' \left\langle g'(X(t)) \times [1 + F'(X(t))(t-t')]\right\rangle \frac{\sigma_0^2}{2\tau_c} \exp\left(-\frac{|t-t'|}{\tau_c}\right)$$
$$\approx \frac{\sigma_0^2}{2}\left[\langle g'(X(t))\rangle + \tau_c \langle g'(X(t))F'(X(t))\rangle\right] + O(\tau_c^2), \tag{9}$$

leading finally to Eq. (5) above (to leading order in τ_c).

Let us show how this works in the simple case $f(X,t) = X^2$, when there is no external force $F(X) = 0$, i.e. $\dot{X} = \eta$. In this case, we know that by construction of the Brownian motion, $\langle X^2(t)\rangle = \sigma_0^2 t$ (see section above). By averaging Eq. (4), one finds:

$$\langle \frac{dX^2}{dt} \rangle = \langle 2X\dot{X}\rangle = \langle 2X\eta\rangle, \tag{10}$$

or, using Eq. (5) above with $g(X) = 2X$ and $g'(X) = 2$,

$$\langle \frac{dX^2}{dt} \rangle = 2\frac{\sigma_0^2}{2} = \sigma_0^2, \tag{11}$$

which recovers the diffusion law $\langle X^2 \rangle = \sigma_0^2 t$.

Itô prescription: $\tau_c \sim \mathrm{d}t$. There is another interpretation of the Langevin equation where Eq. (4) is not correct. It is the limit of the discrete process where the Gaussian noise is independent in each time interval of size $\mathrm{d}t \to 0$. This corresponds to a correlation time τ_c that goes to zero equally fast or faster than the discretization time $\mathrm{d}t$. One explicit realization of Itô's prescription amounts to using a discrete version of the Langevin equation where $X(t+dt) = X(t) + dtF(X(t)) + d\xi(t)$. In this case, $d\xi(t) = \eta(t)\mathrm{d}t$ is chosen independently of all the past $\eta(t')$ and therefore it is also independent of any function of these past $\eta(t')$. In particular $X(t)$ has zero correlation with the "new" $\eta(t)$, so that $\langle g(X(t))\eta(t)\rangle = 0$. So the naive chain rule Eq. (4) would lead, upon averaging, to $\langle dX^2/dt\rangle = 0$! In this case, one has to be very careful with the fact that $d\xi(t)$ is of order $\sqrt{\mathrm{d}t}$, so that to compute correctly df to order $\mathrm{d}t$ one should really write:

$$df = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\mathrm{d}X^2 + \ldots, \tag{12}$$

and using the Langevin equation, $\mathrm{d}X^2 = (\eta + F)^2\mathrm{d}t^2 \approx \eta^2\mathrm{d}t^2$ which is of order $\mathrm{d}t$ since η is of order $\mathrm{d}t^{-1/2}$.

Itô's lemma then shows *rigorously* that in the continuum limit, the correct chain rule in this case is:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X}\dot{X} + \frac{\sigma_0^2}{2}\frac{\partial^2 f}{\partial X^2}, \tag{13}$$

where the last term is called the Itô correction term, that comes from the above mechanism. Now, for our simple example with $F(X) = 0$, $f(X,t) = X^2$ and $\dot{X} = \eta$, one clearly recovers the exact result:

$$\langle \frac{dX^2}{dt} \rangle = \langle 2X\eta \rangle + \frac{\sigma_0^2}{2} \times 2 = \sigma_0^2, \tag{14}$$

since now $\langle X\eta \rangle = 0$, as noted above.

Summary. Comparing the two prescriptions, we note that:

- For Stratonovich, $\eta(t) \sim \sigma_0/\sqrt{\tau_c}$ is *not divergent* when $\mathrm{d}t \to 0$. The usual chain rule applies, but $\eta(t)$ is correlated with $X(t)$ and any function $g(X(t))$.

- For Itô, $\eta(t) \sim \sigma_0/\sqrt{\mathrm{d}t}$ is *divergent* when $\mathrm{d}t \to 0$. The usual chain rule does not apply, and Itô's correction must be added. However, $\eta(t)$ is uncorrelated with $X(t)$ and any function $g(X(t))$.

These two prescriptions correspond to different physical situations, and can lead to different results. Some examples will be given below. In financial applications, the Itô interpretation is more natural since it corresponds to a "noise" (i.e. a price move) that cannot be anticipated.

Question 1: Assuming $X(t)$ follows the Langevin equation with $F(X) = 0$ and an arbitrary correlation time τ_c, compute exactly $\langle X^2(t)\rangle$ for $X(t = 0) = 0$. What happens in the limit $t \gg \tau_c$?

Question 2: Prove the relation Eq. (6.

THE FOKKER-PLANCK EQUATION

Homogeneous case

The Langevin equation is a stochastic differential equation giving the evolution of the position of a Brownian particle, $X(t)$ (but it can also model other quantities, such as the price of a financial asset). What can one infer about the evolution of the probability density $P(X,t|X_0,t=0)$ to observe a certain X at time T, knowing that $X = 0$ (say) at $t = 0$?

One would like to derive a partial differential equation obeyed by $P(X,t|X_0,t=0)$. The trick is to introduce a time independent test function $f(X)$ and study the time evolution of $\langle f \rangle_t$. By definition,

$$\langle f \rangle_t \equiv \int_{-\infty}^{+\infty} \mathrm{d}X P(X,t|X_0,t=0) f(X), \tag{15}$$

from which one gets:

$$\frac{d\langle f\rangle_t}{dt} = \int_{-\infty}^{+\infty} \mathrm{d}X \frac{\partial P(X,t|X_0,t=0)}{\partial t} f(X), \tag{16}$$

Now take a trajectory point of view, using first the Stratonovitch prescription and the chain rule, Eq. (4). Then, upon averaging the change of f during $\mathrm{d}t$,

$$\langle\frac{df}{dt}\rangle = \langle\frac{\partial f}{\partial X}\dot{X}(t)\rangle = \langle\frac{\partial f}{\partial X}[F(X)+\eta(t)]\rangle, \tag{17}$$

which, according to the Stratonovich rule, Eq. (5), reads:

$$\langle\frac{df}{dt}\rangle = \langle\frac{\partial f}{\partial X}F(X)\rangle + \frac{\sigma_0^2}{2}\langle\frac{\partial^2 f}{\partial X^2}\rangle. \tag{18}$$

Making integration by parts, this can be explicitly written as:

$$\langle\frac{df}{dt}\rangle = -\int_{-\infty}^{+\infty} \mathrm{d}X \frac{\partial F(X)P(X,t|X_0,t=0)}{\partial X} f(X) + \frac{\sigma_0^2}{2}\int_{-\infty}^{+\infty} \mathrm{d}X \frac{\partial^2 P(X,t|X_0,t=0)}{\partial X^2} f(X).$$

Comparing Eqs. (16,19), which must be valid for an arbitrary $f(X)$, one obtains the partial differential equation we are looking for, called the '*Fokker-Planck* equation':

$$\frac{\partial P(X,t|X_0,t=0)}{\partial t} = -\frac{\partial}{\partial X}[F(X)P(X,t|X_0,t=0)] + \frac{\sigma_0^2}{2}\frac{\partial^2}{\partial X^2}P(X,t|X_0,t=0), \tag{19}$$

supplemented by the boundary condition: $P(X,t=0|X_0,t=0) = \delta(X-X_0)$. It is easy to show that the same equation is obtained using the Itô prescription.

Among the important properties of Fokker-Planck equation is that it admits the Gibbs-Boltzmann weight as an equilibrium solution. Writing $F(X) = -\partial U/\partial X$ and choosing $\sigma_0^2 = 2k_B T$, it is easy to check that:

$$P_B(X) = \frac{1}{Z}\exp[-\frac{U(X)}{k_B T}] \tag{20}$$

is such that the Fokker-Planck equation is satisfied by a time independent probability, provided the normalisation Z is convergent, i.e.:

$$Z = \int_{-\infty}^{+\infty} \mathrm{d}X \exp[-\frac{U(X)}{k_B T}] < +\infty, \tag{21}$$

i.e. that the potential $U(X)$ grows sufficiently fast when $X \to \pm\infty$ for the particle to be confined. If $Z = \infty$, equilibrium is never reached, the probability distribution keeps spreading to larger and larger distances when $t \to \infty$ (like for example in the free diffusion case, $F(X) \equiv 0$.) In higher dimensions, the force $\vec{F}(\vec{X})$ is not necessarily a gradient. If it is, $\vec{F}(\vec{X}) = -\vec{\nabla} U(X)$, the Boltzmann weight $\exp[-\frac{U(X)}{k_B T}]$ is again an equilibrium solution, provided it can be normalised. If on the other hand, $\vec{F}(\vec{X})$ is a rotational, then $P_S(\vec{X}) = V^{-1}$ uniform within the box of volume V that defines the system is the stationary solution. In the general case, one does not know the form of the stationary solution.

Inhomogeneous diffusion and 'multiplicative' noise

Consider now the case where the variance of the Langevin noise depends explicitly on the position X. The Langevin equation reads:

$$\dot{X}(t) = F(X) + G(X)\eta(t), \tag{22}$$

where $G(X)$ encodes the inhomogeneity of the diffusion constant. Now, the variable X and the noise η appear in a multiplicative fashion, and the microscopic specification of the model will be crucial. Repeating the very same calculation as above in the one dimensional case, one finds that the Fokker-Planck equation takes different forms depending on whether Itô's prescription or Stratonovitch's prescription is used. Writing $D(X) = \sigma_0^2 G^2(X)/2$, one finds in the former case,:

$$\begin{aligned}\frac{\partial P(X,t|X_0,t=0)}{\partial t} &= -\frac{\partial}{\partial X}[F(X)P(X,t|X_0,t=0)] \\ &+ \frac{\partial^2}{\partial X^2}[D(X)P(X,t|X_0,t=0)],\end{aligned}$$

whereas in the latter case,

$$\begin{aligned}\frac{\partial P(X,t|X_0,t=0)}{\partial t} &= -\frac{\partial}{\partial X}[F(X)P(X,t|X_0,t=0)] \\ &+ \frac{\partial}{\partial X}\left[\sqrt{D(X)}\frac{\partial}{\partial X}\sqrt{D(X)}P(X,t|X_0,t=0)\right].\end{aligned}$$

So in this case one sees that the two prescriptions do *not* lead to the same evolution of $P(X,t|X_0,t=0)$.

Question 3: Compute the equilibrium state of the Fokker-Planck equation for $F(X) = -kX$ and $G(X) = X$, for both prescriptions (Itô and Stratonovich).