ICFP M2 - Statistical physics: Advanced and New Applications

TD4: Mean-Field Theory
Large dimensional limit and Bethe lattices

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Mean-field theory is a very powerful tool with several applications in physics. It also had a strong impact in other fields, such as for example computer science, information and probability theory. Today we are going to study its foundation, which is twofold: on one hand it is related to the infinite dimensional limit and on the other it is related to exactly solvable models defined on special kinds of lattices (often called mean-field like for this reason).

1 Mean-Field Theory and the Infinite Dimensional Limit

We focus on the free energy as a function of the magnetization, or Landau free-energy, defined as:

$$-\beta F(\{m_i\}) = \ln \sum_{\{S_i\}} \exp\left(-\beta \sum_{\langle ij\rangle} J_{ij} S_i S_j + \sum_i \beta h_i (S_i - m_i)\right)$$

The h_is are functions of the m_is and β set by the equations

$$\langle S_i \rangle = m_i$$

where the average is performed with the Hamiltonian above in presence of the magnetic field.
The Ising model we focus on sits on a hypercubic lattice in d spatial dimensions, the coupling constant $J_{ij} = -\frac{1}{2d}$ among nearest neighbors spins on the lattice and zero otherwise.

We are now going to perform an expansion introducing the parameter α

$$-\beta F(\{m_i\}, \alpha) = \ln \sum_{\{S_i\}} \exp\left(-\alpha\beta \sum_{\langle ij\rangle} J_{ij} S_i S_j + \sum_i \beta h_i (S_i - m_i)\right)\Bigg|_{\alpha=1}$$

The expansion is in powers of α, and α is set equal to one at the end:

$$-\beta F(\{m_i\}, \alpha) = -\beta F(\{m_i\}, 0) + \frac{\partial}{\partial \alpha}[-\beta F(\{m_i\}, \alpha)]\Bigg|_{\alpha=0} \alpha + \frac{1}{2}\frac{\partial^2}{\partial \alpha^2}[-\beta F(\{m_i\}, \alpha)]\Bigg|_{\alpha=0} \alpha^2 + \cdots$$

As we shall see, expanding in α corresponds to a $1/d$ expansion.

1. Compute $-\beta F(\{m_i\}, 0)$. In order to do this, first find the relationship between h_i and m_i valid for $\alpha = 0$ and then invert it and use the definition of $F(\{m_i\}, 0)$.

$$-\beta F(\{m_i\}, 0) = \sum_i \left[-\frac{1-m_i}{2} \ln \frac{1-m_i}{2} - \frac{1+m_i}{2} \ln \frac{1+m_i}{2}\right]$$

   As a check, set $m_i = 0$ and explain why the resulting value is the correct one.

2. Show that

$$\frac{\partial}{\partial \alpha}[-\beta F(\{m_i\}, \alpha)]\Bigg|_{\alpha=0} = -\beta \sum_{\langle ij\rangle} J_{ij} m_i m_j$$

3. The expansion in α can be continued straightforwardly (but a bit painfully at higher order). The second term reads:

$$\frac{1}{2}\frac{\partial^2}{\partial\alpha^2}[-\beta F(\{m_i\},\alpha)]\Big|_{\alpha=0} = \frac{\beta^2}{2}\sum_{\langle ij\rangle} J_{ij}^2(1-m_i^2)(1-m_j^2)$$

By evaluating the free energy for a uniform magnetization profile $m_i = m$ show that this second term is negligible in the large d limit.

4. Using the first three terms in the expansion in α (and now setting $\alpha = 1$) obtain the equations verified by the magnetization in zero field:

$$m_i = \tanh\left[-\sum_{j(\neq i)}\beta J_{ij}m_j - \beta^2\sum_{j(\neq i)} J_{ij}^2 m_i(1-m_j^2)\right]$$

Focus on homogenous solutions $m_i = m$ (one can show that the solution minimizing the free-energy is homogeneous) and, by using explicitly that the coupling constant $J_{ij} = -\frac{1}{2d}$ among nearest neighbors spins on the lattice and zero otherwise, show that the usual mean-field equations correspond to the first order in the $1/d$ expansion and, in consequence, they are exact in the infinite dimensional limit.

5. Let's do now an exact Landau expansion in zero field by considering a uniform magnetization profile $m_i = m$ and defining:

$$-\beta F(\{m_i\}/N = A_0(\beta) + A_1(\beta)m^2 + A_2(\beta)m^4 + O(m^6)$$

Show that $A_1(\beta) = \frac{\beta}{2} - \frac{1}{2} - \frac{\beta^2}{4d} + \cdots$

6. $A_2(\beta)$ can be also computed: $A_2(\beta) = -\frac{1}{12} + \frac{\beta^2}{8d} + \cdots$. Using these expressions and your knowledge of Landau theory compute the critical β_c for $d \to \infty$ and its first correction in $1/d$. For comparison, numerical simulations give $\beta_c(3D) \simeq 1.3272$, you can check how the $1/d$ expansion performs.

7. The mean field approximation is exact for the Ising model defined on a completely connected lattice, i.e. where each spin interact with all other spins by a coupling $-1/N$. This is called the Curie-Weiss model. Using the expansion we worked out, show that only the first two contributions to the free energy have to be retained in this case and hence the mean-field equations are exact for the Curie-Weiss model.

8. Starting from the first two contributions to $-\beta F(\{m_i\})$, take the continuum limit and write down the expression of the Landau free energy obtained from the $1/d$ expansion.

2 Bethe approximation and random regular graphs (exercise for later)

Another mean-field approximation that is often used is the so-called Bethe approximation. It is obtained neglecting loops, or replacing the local euclidean structure by a tree-like one, see Fig.1. In the following we will apply to the Ising model in two dimensions (just to keep the notation simple). To be consistent with the notation in the figure we will denote now the Ising spins with the greek letter σ.

1. Due to the tree-like structure the probability law of $\sigma_1, \sigma_2, \sigma_3$ in absence of σ_0 (the spin in the center) is factorized:

$$p_0(\sigma_1,\sigma_2,\sigma_3) = p_{1\to 0}(\sigma_1)p_{2\to 0}(\sigma_2)p_{3\to 0}(\sigma_3)$$

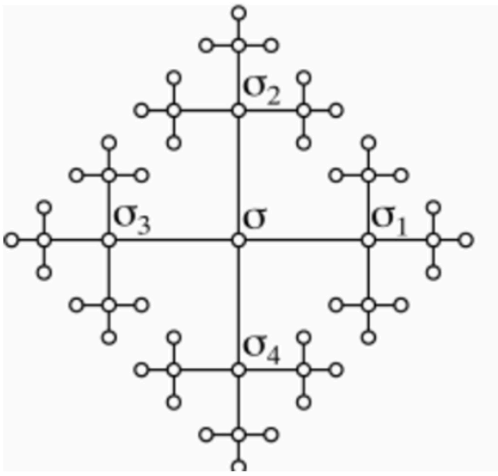

Figure 1: Tree-like Bethe approximation of a square lattice (the spins are called σ in the figure).

where $p_{1\to 0}(\sigma_1)$ is the marginal probability distribution of σ_1 in absence of σ_0. For a binary variable $\sigma_i = \pm 1$ justify why the probability law can be written in terms of an effective field h_i:

$$p_{i\to 0}(\sigma_i) = \frac{e^{\beta h_i \sigma_i}}{2\cosh(\beta h_i)}$$

2. Starting from the recursive equation:

$$p_{0\to 4}(\sigma_0) = \mathcal{N} \sum_{\sigma_1,\sigma_2,\sigma_3} p_{1\to 0}(\sigma_1) p_{2\to 0}(\sigma_2) p_{3\to 0}(\sigma_3) \prod_{i=1}^{3} e^{\beta J \sigma_0 \sigma_i}$$

where $\mathcal{N}$ is a normalization constant, obtain the recursive equation on the effective fields:

$$\beta h_0 = \sum_i \tanh^{-1}[\tanh(\beta h_i)\tanh(\beta J)]$$

3. Considering a uniform field profile and defining $\tilde{m} = \tanh(\beta h)$, write the mean-field equation on $\tilde{m}$.

4. Generalizing the previous equation to a hypercubic lattice in d dimensions and setting $J = 1/(2d)$ obtain β_c in the limit $d \to \infty$ and its first correction in $1/d$.

As the previous mean-field approximation is exact for the Curie-Weiss model, one can show that the Bethe approximation is exact on a random-(2d) regular graph, which is defined as a graph taken at random with uniform measure on the set of graphs with the same connectivity $2d$ for each site.
The Bethe approximation has had an enormous success and very broad applications in computer science and information theory in the last decades. The main reason is that in these fields one often encounter problems defined on networks that resemble random regular graphs or have a local tree like structure, and on which the Bethe approximation is exact.

2.1 Solution

1. The probability distribution for a binary variable σ is fixed once $p(\sigma = +1)$. The representation

$$p(\sigma_i) = \frac{e^{\beta h_i \sigma_i}}{2\cosh(\beta h_i)}$$

fixes $p(\sigma_i = +1)$ through h_i and is normalized, i.e. it verifies $p(\sigma_i = -1) = 1 - p(\sigma_i = +1)$. In consequence it is a general representation of $p(\sigma_i)$.

2. The probability of observing the spin σ_0 in absence of spin σ_4 is obtained by summing over all configurations but the spins $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ using the Boltzmann weight. This leads to the equation

$$p_{0\to 4}(\sigma_0) = \mathcal{N} \sum_{\sigma_1,\sigma_2,\sigma_3} p_{1\to 0}(\sigma_1)p_{2\to 0}(\sigma_2)p_{3\to 0}(\sigma_3) \prod_{i=1}^{3} e^{\beta J \sigma_0 \sigma_i}$$

   Using that

$$\sum_{\sigma_i} p_{i\to 0}(\sigma_i) e^{\beta J \sigma_0 \sigma_i} = \mathcal{N}' e^{\sigma_0 \tanh^{-1}[\tanh(\beta h_i)\tanh(\beta J)]}$$

   and multiplying together the three contributions from $\sigma_1, \sigma_2, \sigma_3$, one finds the recurrence equation on the effective fields h_i.

3. Taking the hyperbolic tangent of the recursive equation on the effective fields for a uniform field configuration one obtains:

$$\tilde{m} = \tanh[3\tanh^{-1}[\tilde{m}\tanh(\beta J)]]$$

4. In order to generalize to d dimension, one has to consider that there are $2d$ neighbors; excluding the "outgoing" spin (the counterpart of σ_4) one gets:

$$\tilde{m} = \tanh[(2d-1)\tanh^{-1}[\tilde{m}\tanh(\beta/2d)]]$$

5. This equation can be studied graphically. It admits a solution with $\tilde{m}$ different from zero when the right hand side starts at small m with a slope larger than one. The equation for the critical temperature is therefore:

$$1 = (2d-1)\tanh(\beta_c/2d)$$

   which gives $\beta_c = 1 + \frac{1}{2d} + O(1/d^2)$ in the large d limit.

Note that $\tilde{m}$ is not the true magnetization, since it is obtained by cutting one interaction among the $2d-1$ possible ones. The true magnetization is instead $m = \tanh[2d\tanh^{-1}[\tilde{m}\tanh(\beta/2d)]]$. When $\tilde{m}$ becomes different from zero, so does m.