GENERAL RELATIVITY

M2 Theoretical physics

December 5, 2018.

1 Cartan formalism

Reminder
Consider one p-form Θ^1 and one q-form Θ^2, then $\Theta^1 \wedge \Theta^2$ is a $p+q$-form. It acts on vectors as follows

$$\Theta^1 \wedge \Theta^2 \left(u_1 \dots u_{p+q}\right) = \sum_\sigma \epsilon\left(\sigma\right) \Theta^1 \left(u_{\sigma(1)} \dots u_{\sigma(p)}\right) \Theta^2 \left(u_{\sigma(p+1)} \dots u_{\sigma(p+q)}\right)$$

where σ is a permutation of $[1, p+k]$ and $\epsilon\left(\sigma\right)$ its signature.
For $1-$forms λ and β, it implies that $\lambda \wedge \alpha = -\alpha \wedge \beta$.
The differential operator acts on such product as :

$$\mathrm{d}\left(\alpha\Theta^1 \wedge \dots \wedge \Theta^p\right) = \frac{\partial \alpha}{\partial x^i} \mathrm{d}x^i \wedge \Theta^1 \wedge \dots \wedge \Theta^p + \alpha \mathrm{d}\Theta^1 \wedge \Theta^2 \dots \wedge \Theta^p + \dots$$

We recall that for any function f one has $\mathrm{d}\left(\mathrm{d}f\right) = 0$.
Cartan's method proceeds as follows :

- Define a triad Θ^α such that $g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}\Theta^\mu\Theta^\nu$, where $\eta_{\mu\nu}$ is Minkowski's metric (so that all the indices are hereafter manipulated by η).
- Compute the connection $1-$ forms w^μ_ν using
  - the first equation of structure $\mathrm{d}\Theta^\mu + w^\mu_\nu \wedge \Theta^\nu = 0$.
  - the metricity condition $w_{\mu\nu} + w_{\nu\mu} = 0$.
- Compute the curvature 2-forms $\mathcal{R}^\mu_\nu$ using the second equation of structure

$$\mathcal{R}^\mu_\nu = \mathrm{d}w^\mu_\nu + w^\mu_\alpha \wedge w^\alpha_\nu.$$

- Read from $\mathcal{R}$ Riemann tensor from

$$\mathcal{R}^\mu_\nu = \frac{1}{2} R^\mu_{\nu\alpha\beta} \Theta^\alpha \wedge \Theta^\beta.$$

Application to Schwarzschild metric
Consider Schwarzschild metric written as

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + e^{2\lambda}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)$$

where ν and λ are purely radial functions.
1-a) Find a proper choice of triad Θ^α.

1-b) Translate the metricity conditions into conditions on ${w^\mu}_\nu$. What do they imply for $\mathcal{R}^\mu_\nu$ and for $R^\mu_{\nu\alpha\beta}$?
1-c) Find the connections 1−forms.
1-d) Compute the curvature 2−forms.
1-e) Deduce the value of the Riemann tensor.
1-f) For Schwarzschild solution one has

$$e^{-2\lambda} = e^{2\nu} = 1 - \frac{2M}{r}.$$

Show that the Ricci tensor is identically zero.

2 Einstein-Hilbert action

We consider the Einstein-Hilbert action :

$$S = \int_\Omega \sqrt{-g} R \mathrm{d}^4 x$$

where Ω is a 4-dimensional domain to be specified. $\mathbf{g}$ is the 4-dimensional metric, R its associated Ricci scalar and the covariant derivative will be denoted by ∇.
2-a) Show that the variation of the Ricci tensor can be written as

$$\delta R_{\mu\nu} = \nabla_\alpha \left(\delta \Gamma^\alpha_{\mu\nu}\right) - \nabla_\nu \left(\delta \Gamma^\alpha_{\alpha\mu}\right).$$

2-b) Show that the variation of the Ricci scalar is

$$\delta R = \nabla_\alpha V^\alpha + R_{\mu\nu} \delta g^{\mu\nu}.$$

Find the expression for V^α
Thanks to Gauss integral formula, one can show that the term involving V^α is a surface term. It will be studied later (see **3)**) and one can discard it for now...
2-c) Show then that the variation of the action is

$$\delta S = \int_\Omega \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} \mathrm{d}^4 x$$

where $G_{\mu\nu}$ is Einstein tensor, so that Einstein equations in vacuum are indeed recovered.
We recall that Jacobi's formula: $g_{\mu\nu}\delta g^{\mu\nu} = -\dfrac{\delta g}{g}$.

3 Gibbson-Hawking-York boundary term

The application of the Gauss integral formula reads

$$\int_\Omega \sqrt{-g} \nabla_\alpha V^\alpha \mathrm{d}^4 x = \int_{\partial\Omega} \epsilon \sqrt{|h|} V^\alpha N_\alpha \mathrm{d}^3 y$$

where y is a set of coordinates of $\partial\Omega$, $\vec{N}$ is the unit normal such that $N_\mu N^\mu = \epsilon$ and h the induced metric on the surface.
3-a) Show that

$$\delta \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left[\nabla_\mu \delta g_{\beta\nu} + \nabla_\nu \delta g_{\beta\mu} - \nabla_\beta \delta g_{\mu\nu}\right].$$

3-b) Deduce that $V^\lambda = \left(g^{\alpha\lambda}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta}\right)\nabla_\nu \delta g_{\alpha\beta}$.
3-c) Under the assumption that $\delta\mathbf{g} = 0$ on $\partial\Omega$ (i.e. Dirichlet boundary conditions), show that the surface term appearing in δS (i.e. the term involving V^α) can be written as

$$I_{\partial\Omega} = -\int_{\partial\Omega} \epsilon\sqrt{|h|}h^{\alpha\beta}N^\mu\partial_\mu\left(\delta g_{\alpha\beta}\right)\mathrm{d}^3y.$$

hint : one can use $g_{\mu\nu} = h_{\mu\nu} + \epsilon N_\mu N_\nu$.
3-d) In order to cancel the term $I_{\partial\Omega}$, one can add a surface integral to Einstein-Hilbert action. The Gibbson-Hawking-York term is a valid choice and is given by

$$S_{\mathrm{GHY}} = 2\int_{\partial\Omega}\epsilon\sqrt{|h|}Y\mathrm{d}^3y$$

where Y is the trace of the extrinsic curvature tensor of $\partial\Omega$. One will use the convention such that $Y = h^{\alpha\beta}\nabla_\alpha N_\beta$.
Show that the variation δS_{GHY} does indeed cancel the term $I_{\partial\Omega}$ when $\delta\mathbf{g} = 0$ on $\partial\Omega$. One will first show that $\delta N_\mu = -1/2\epsilon N_\mu N_\alpha N_\beta \delta g^{\alpha\beta}$.

4 Hamiltonian formulation

One wishes to make the link between the 3+1 formalism and the Hamiltonian description of the theory (following the work of Arnowitt, Deser and Misner). One will consider only the vacuum case. Forgetting, for now, the GHY term and all other boundary terms, the action of the theory is $S = \int \ {}^4R\sqrt{-g}\mathrm{d}^4x$.
Using the 3+1 quantities it can be rewritten as

$$S = \int_t\left\{\int_{\Sigma_t} L\mathrm{d}^3x\right\}\mathrm{d}t = \int_t\left\{\int_{\Sigma_t} N\left(R + K_{ij}K^{ij} - K^2\right)\sqrt{\gamma}\mathrm{d}^3x\right\}\mathrm{d}t.$$

4-a) The variables of this Lagrangian density L are the $q = (N, B^i, \gamma_{ij})$. Compute K_{ij} as a function of those variables and their time derivative. Deduce the various conjugate momenta $\pi = \dfrac{\partial L}{\partial \dot{q}}$.
4-b) Using the usual Legendre transform, compute the Hamiltonian density $\mathcal{H} = \sum \pi\dot{q} - L$. One will write the result as

$$H = -\int_{\Sigma_t}\left(NC_0 - 2B^iC_i\right)\sqrt{\gamma}\mathrm{d}x^3.$$

4-c) What are Hamilton's equations associated with N and B^i ? After some more complex computations, one can show that the equations associated with π^{ij} and γ_{ij} give the definition of K_{ij} and its evolution equation.

5 REMINDER the 3+1 formalism

We recall that in the 3+1 formalism, spacetime is described as a family of spatial hypersurfaces on which purely spatial quantities are defined. The whole spacetime is obtained by giving the time evolution of those quantities.

Each hypersurface is defined by its normalized normal n^μ. The extrinsic curvature tensor K_{ij} measures the variation of the normal on each hypersurface.
The induced metric on the hypersurfaces is then $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. For spatial tensors (i.e. tensors such that $n_\mu V^\mu = 0$), one can relate the 4D derivative to the 3D ones by using the projection operator γ. For instance, one has

$$D_\mu V^\nu = \gamma_\mu^\alpha \gamma_\beta^\nu \nabla_\alpha V^\beta.$$

The metric is then written as

$$\mathrm{d}s^2 = -\left(N^2 - B_i B^i\right) \mathrm{d}t^2 + 2B_i \mathrm{d}t \mathrm{d}x^i + \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j,$$

where N is the lapse, $\vec{B}$ the shift and γ_{ij} the induced metric. The normal is given by $n^\alpha = \left(\frac{1}{N}, -\frac{B^i}{N}\right)$.
Einstein's equations are written as a first order evolution system (here in vacuum)

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathcal{L}_{\vec{B}}\right) \gamma_{ij} &= -2NK_{ij} \\ \left(\frac{\partial}{\partial t} - \mathcal{L}_{\vec{B}}\right) K_{ij} &= -D_i D_j N + N\left(R_{ij} + KK_{ij} - 2K_{ik}K_j^k\right). \end{aligned}$$

Those evolution equations are supplemented by the constraints

- Hamiltonian constraint $R + K^2 - K_{ij}K^{ij} = 0$.
- momentum constraint $D_j K_i^j - D_i K = 0$.