ICFP M2 - Statistical physics 1 – TD n° 5
Migdal-Kadanoff bond moving approximation

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In this tutorial, we consider the Ising model on a d-dimensional square lattice defined by the partition function

$$Z = \sum_{\{\sigma_i\}} \exp\left[K \sum_{\langle i,j \rangle} \sigma_i \sigma_j\right] \tag{1}$$

where $\sum_{\langle i,j \rangle}$ denotes a sum over nearest neighbours, $\sigma_i = \pm 1$ are spin variables and $K > 0$ is the ferromagnetic coupling (multiplied by the inverse temperature β). The real-space renormalisation group can be performed exactly on the $1d$ Ising model by decimating a regular sequence of spins. This yields the following recursion relation for the coupling constant

$$K' = \frac{1}{2} \ln \cosh(2K) \; . \tag{2}$$

Unfortunately, this decimation can not be carried out exactly in higher dimensions and some truncation approximations are necessary. One such scheme, which proved to be quite versatile is the so-called *Migdal-Kadanoff renormalisation* [1, 2], which we illustrate here on the Ising model on the d-dimensional square lattice defined in (1) – see also [3, 4] for further applications.

1 Migdal-Kadanoff real space RG on the $2d$-Ising model

1. Suppose first that we want to perform a decimation by dividing the $2d$-lattice into two sub-lattices. As in the $1d$ case we start by decimating the spins on one sub-lattice (see Fig. 1). Show that the interaction between the four spins surrounding each decimated spin is

$$R(\sigma_1', \sigma_2', \sigma_3', \sigma_4') = 2\cosh\left(K(\sigma_1' + \sigma_2' + \sigma_3' + \sigma_4')\right) \; . \tag{3}$$

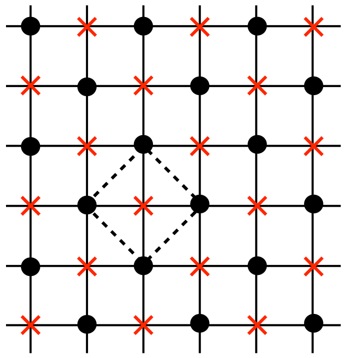

Figure 1: Standard decimation procedure of a single spin in the $2d$-Ising model.

2. Show that R in Eq. (3) can be rewritten as

$$R = \exp\left(g' + K'(\sigma_1'\sigma_2' + \sigma_1'\sigma_3' + \sigma_1'\sigma_4' + \sigma_2'\sigma_3' + \sigma_2'\sigma_4' + \sigma_3'\sigma_4') + K_4'\sigma_1'\sigma_2'\sigma_3'\sigma_4'\right) \; . \tag{4}$$

What do you conclude from this ?

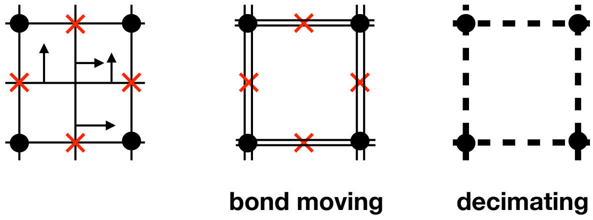

Figure 2: Illustration of the Migdal-Kadanoff procedure of the $2d$-Ising model.

An alternative strategy (see Fig. 2) is to decimate every other line or column of spins along each lattice direction but removing the bonds not connected to the retained spins. Obviously, this simple approximation weakens the whole system, making it more "one-dimensional". This is remedied by using the unwanted bonds to reinforce those that are left behind: the spins that are retained are now connected by a pair of double bonds of strength $2K$ (see Fig. 2).

3. Show that the decimation rule reads

$$K' = \frac{1}{2} \ln \cosh(4\,K)\ . \tag{5}$$

4. What is the scale factor $b > 1$ between the system with coupling K and the one with coupling K' ?

5. What are the fixed points of this recursion relation (5)? Analyse their stability and compare this with the $1d$ case. Show that the Migdal-Kadanoff approximation predicts a phase transition for $K = K^*$ (it should be compared to the exact value of the critical coupling $K_c \approx 0.441$).

6. By linearizing the recursion relation (5) near $K = K^*$, compute the critical exponent ν.

2 Migdal-Kadanoff real space RG for the d-dimensional Ising model

We now consider the Ising model on the d-dimensional on the square lattice (1).

1. Show that the recursion relation reads

$$K' = \frac{1}{2} \ln \cosh(2^d K)\ . \tag{6}$$

2. Identify the fixed points of this recursion (6) and analyse their stability.

3. What is the lower critical dimension predicted by the Migdal-Kadanoff RG ? Compare this result to the exact result.

4. What about the upper-critical dimension ?

5. What is the value of the exponent ν in d dimensions ?

References

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[3] P. M. Chaikin, T. C. Lubensky, *Principles of condensed matter physics*, Cambridge University Press, (1995).

[4] M. Kardar, *Statistical physics of fields*, Cambridge University Press, (2007).