

The nonperturbative renormalization group: the local potential approximation

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The idea of the nonperturbative renormalization group (NPRG) is to implement Wilson’s momentum shell integration by adding to the original partition function a momentum dependent mass term that gives a large “mass” to slow modes while leaving unaffected the rapid ones [1]. For a field theory with one scalar field ϕ and Hamiltonian $H[\phi]$, we thus define a scale dependent family of partition functions, indexed by k , via the path-integral

$$Z_k[B] = \int \mathcal{D}\phi e^{-H[\phi] - \Delta H_k[\phi] + \int_x B \phi}, \quad (1)$$

where we have set $k_B T = 1$ and used the notation $\int_x \equiv \int d^d x$ for the real space integral over the whole volume. We denote by Λ the ultra-violet cutoff, $a = 2\pi/\Lambda$ being the microscopic length scale (i.e. lattice spacing) of the model. In Eq. (1), the supplement of Hamiltonian $\Delta H_k[\phi]$ reads

$$\Delta H_k[\phi] = \frac{1}{2} \int_q \hat{R}_k(q) \hat{\phi}(q) \hat{\phi}(-q), \quad \text{with } 0 \leq k \leq \Lambda, \quad (2)$$

where $\int_q = \int \frac{d^d q}{(2\pi)^d}$ denotes an integral in momentum space and we used the notation \hat{f} to denote the Fourier transform of the function f . In particular, $\hat{\phi}(q) = \int_x e^{iqx} \phi(x)$ (note that x and q are d -dimensional vectors). Here we will use a particular choice of the regulator $\hat{R}_k(q)$

$$\hat{R}_k(q) = (k^2 - q^2) \theta(k^2 - q^2), \quad (3)$$

where $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$, which is convenient for analytical computations.

The NPRG relies on an exact equation of evolution, when k is lowered from $k = \Lambda$ down to $k \rightarrow 0$, for the effective action $\Gamma_k[M]$, which is the Legendre transform of $\ln Z[B]$ in Eq. (1), which reads

$$\partial_k \Gamma_k[M] = \frac{1}{2} \int_{x,y} \partial_k R_k(x-y) \left[\Gamma_k^{(2)} + \mathcal{R}_k \right]^{-1}(x,y) \quad (4)$$

where $\mathcal{R}_k(x,y) = R_k(x-y)$, $\Gamma_k^{(2)}(x,y) = \frac{\delta^2 \Gamma_k[M]}{\delta M(x) \delta M(y)}$ and where the inverse in Eq. (4) is understood in the sense of operators, $\int_y O^{-1}(x,y) O(y,z) = \delta^d(x-z)$. The solution to this equation (4) is then fully specified by the initial condition $\Gamma_{k=\Lambda}[M] = H[M]$, which is the original Hamiltonian (since no fluctuations are integrated out at that scale [see Eq. (3)]).

This equation (4) is exact (it contains no approximation) and very general (it holds in principle for any field theory) but it turns out that solving it, to extract for instance critical exponents, is extremely hard. The goal of this tutorial is to present an approximation scheme, the so called “Local Potential Approximation” (LPA) which has been widely used in the literature and contains most of the ideas that guided the most recent developments. We will then apply it to the Ising model.

1 Local Potential Approximation: general framework

The idea of the LPA is to propose an ansatz for the (approximate) solution of Eq. (4) of the form

$$\Gamma_k[M] = \int_x \left[\frac{1}{2} (\nabla_x M(x))^2 + U_k[M(x)] \right]. \quad (5)$$

1. Let $M(x) = M_{\text{unif}}$ be a uniform configuration. Show that

$$U_k(M_{\text{unif}}) = \frac{1}{V} \Gamma_k[M_{\text{unif}}], \quad (6)$$

where V is the volume of the system.

2. Let $O(x, y) \equiv O(x - y)$ a translation invariant operator, such that $O(x) = O(-x)$, show that its inverse $O^{-1}(x, y)$ can be computed via the use of Fourier transform, i.e.

$$O^{-1}(x, y) = \int_q e^{iq(x-y)} \frac{1}{\hat{O}(q)}, \quad \hat{O}(q) = \int_x O(x) e^{iqx}. \quad (7)$$

Hint: note that $\psi_q(y) = \frac{1}{(2\pi)^{d/2}} e^{iqy}$ is an eigenvector of $O(x, y)$.

3. Deduce from Eq. (7) that the RG equation for $U_k(M)$ reads

$$\partial_k U_k(M) = \frac{1}{2} \int_q \frac{\partial_k \hat{R}_k(q)}{q^2 + \hat{R}_k(q) + \frac{\partial^2}{\partial M^2} U_k(M)}. \quad (8)$$

4. For the specific choice of the cut-off function in Eq. (3) show that the integral over q can be performed explicitly, yielding

$$\partial_k U_k(M) = \frac{4v_d}{d} \frac{k^{d+1}}{k^2 + \frac{\partial^2}{\partial M^2} U_k(M)} \quad (9)$$

where $v_d = 1/(2^{d+1} \pi^{d/2} \Gamma(d/2))$. We recall that the area of the unit sphere in d -dimensions is $S_d = 2\pi^{d/2} / \Gamma(d/2)$.

2 Application to the Ising model

In this case, one expects that the effective action $\Gamma_k[M]$ is invariant under the symmetry group \mathbb{Z}_2 . Hence, we assume that

$$U_k(M) = u_k \left(\rho = \frac{M^2}{2} \right). \quad (10)$$

1. Show that the equation for $u_k(\rho)$ reads

$$\partial_k u_k(\rho) = \frac{4v_d}{d} \frac{k^{d+1}}{k^2 + u'_k(\rho) + 2\rho u''_k(\rho)}. \quad (11)$$

2. In order to find fixed-point solutions to this RG equation (11) one has to work with dimensionless variables. Since k has dimension of an inverse length (see for instance Eq. (3)), justify that the dimension of Γ_k is zero, i.e. $[\Gamma_k] = k^0$, which in turn implies that the dimensions of M and U_k are

$$[M] = k^{(d-2)/2}, \quad [U_k] = [u_k] = k^d. \quad (12)$$

We thus define the dimensionless quantities

$$\tilde{x} = kx, \quad \tilde{\rho}(\tilde{x}) = k^{2-d} \rho(x), \quad \tilde{u}_t(\tilde{\rho}) = k^{-d} u_k(\rho), \quad \text{with } t = \ln(k/\Lambda). \quad (13)$$

3. Obtain the RG equation for the rescaled potential \tilde{u}_t under the form

$$\partial_t \tilde{u}_t = -d \tilde{u}_t + (d-2) \tilde{\rho} \tilde{u}'_t + \frac{4v_d}{d} \frac{1}{1 + \tilde{u}'_t + 2\tilde{\rho} \tilde{u}''_t}. \quad (14)$$

In general, the study of this RG equation (the right hand side is the so-called β -function) is quite difficult and has to be handled numerically. Here, to get some insights on this equation, we expand $\tilde{u}_t(\tilde{\rho})$ in powers of $\tilde{\rho}$, up to order $O(\tilde{\rho}^2) = O(M^4)$ (in the spirit of the Ginzburg-Landau ϕ^4 theory) and we write

$$\tilde{u}_t(\tilde{\rho}) = \frac{g_t}{2} (\tilde{\rho} - \mu_t)^2, \quad (15)$$

and we assume that, at scale $k = \Lambda$, $\mu_0 > 0$ – which corresponds to a symmetry broken phase with a spontaneous magnetization $M_{\text{sp}} = \Lambda^{(d-2)/2} \sqrt{2\mu_0}$ at the mean field level of approximation (i.e., when fluctuations are not taken into account). However, the integration of the fluctuations can modify profoundly this picture: as we will see the minimum of the potential μ_t has a non-trivial RG flow that can drive it to zero.

To derive the RG equations of μ_t and g_t from (15), it is useful to define μ_t and g_t as

$$\left. \frac{\partial \tilde{u}_t}{\partial \tilde{\rho}} \right|_{\tilde{\rho}=\mu_t} = 0 \quad (16)$$

$$\left. \frac{\partial^2 \tilde{u}_t}{\partial \tilde{\rho}^2} \right|_{\tilde{\rho}=\mu_t} = g_t. \quad (17)$$

4. The flow equation for λ_t and μ_t can be obtained by taking derivatives of these equations (16) and (17) with respect to t . Note that $\tilde{u}_t(\kappa_t)$ depends on t (i) explicitly through the t -dependence of the function \tilde{u}_t and (ii) implicitly through the t -dependence of the argument κ_t where it is evaluated. Based on this, show that

$$\frac{\partial}{\partial t} \left[\left. \frac{\partial^n \tilde{u}_t}{\partial \tilde{\rho}^n} \right|_{\tilde{\rho}=\mu_t} \right] = \left. \frac{\partial^n \partial_t \tilde{u}_t}{\partial \tilde{\rho}^n} \right|_{\tilde{\rho}=\mu_t} + \partial_t \kappa_t \left. \frac{\partial^{n+1} \tilde{u}_t}{\partial \tilde{\rho}^{n+1}} \right|_{\tilde{\rho}=\mu_t} \quad (18)$$

and obtain the following RG equations for μ_t and g_t

$$\partial_t \mu_t = -(d-2)\mu_t + \frac{12v_d}{d} \frac{1}{(1 + 2g_t \mu_t)^2} \quad (19)$$

$$\partial_t g_t = (d-4)g_t + \frac{72v_d}{d} \frac{g_t^2}{(1 + 2g_t \mu_t)^3}. \quad (20)$$

5. Discuss qualitatively these equations (19) and in particular the three possible scenarios: (i) the system is in its broken symmetry phase, (ii) the system is in the high temperature phase and (iii) the system is critical.

3 Study of the fixed point around the dimension $d = 4$

1. Argue that $d = 4$ is the *upper critical dimension* for this system, i.e., that the coupling constant g_t is irrelevant in $d > 4$.

2. We see the these RG equations (19) can be defined for any value of d , even non-integer ones. We thus set $d = 4 - \epsilon$ and consider the limit $\epsilon \ll 1$. In this limit, show that the RG equations read (up to order 2 in the parameters g_t and μ_t)

$$\partial_t \mu_t = \beta(\mu_t) = -(2 - \epsilon)\mu_t + \frac{3}{32\pi^2} - \frac{3}{8\pi^2} g_t \mu_t \quad (21)$$

$$\partial_t g_t = \beta(g_t) = -\epsilon g_t + \frac{9}{16\pi^2} g_t^2 . \quad (22)$$

3. Show that these equations admit a non-trivial fixed point solution (μ^*, λ^*) , known as the Wilson-Fisher fixed point [3].
4. The critical exponents characterize the behavior of the RG flow around the fixed point. In particular, the exponent γ characterizing the divergence of the magnetic susceptibility close to T_c can be obtained from the eigenvalue λ characterizing the unstable direction around the RG fixed point, i.e.

$$\partial_t(\mu_t - \mu^*) \simeq -\lambda(\mu_t - \mu^*) , \text{ and } \gamma = \frac{2}{\lambda} . \quad (23)$$

Show that $\gamma = 1 + \epsilon/6 + o(\epsilon)$. At this order in ϵ , one can check that this result for the exponent λ , using LPA, coincides with the perturbative ϵ expansion (at one-loop order). However, LPA fails to reproduce the correct result of the ϵ expansion at order $O(\epsilon^2)$ (i.e. at two-loop order), which shows explicitly that this is an approximation.

References

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