ICFP M2 - Statistical physics 2 – TD n° 7
Wigner semi-circle law for random matrices

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We consider in this problem random $N \times N$ real symmetric matrices M, their eigenvalues being denoted $\lambda_1, \dots, \lambda_N$, and investigate the limiting behavior of their empirical eigenvalue distribution $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ when N diverges.

One says that M is drawn from the Gaussian Orthogonal Ensemble (GOE) when its matrix elements M_{ij} above the diagonal ($i \leq j$) are independent Gaussian random variables with zero mean and variance $\mathbb{E}[M_{ij}^2] = \frac{1}{N}$ for $i < j$ and $\mathbb{E}[M_{ii}^2] = \frac{2}{N}$ on the diagonal. The elements M_{ij} below the diagonal ($i > j$) are then obtained from the symmetry $M_{ij} = M_{ji}$.

One calls real Wigner matrix a generalization of the GOE, with the same assumptions of independence, symmetry, values of the first and second moment of the M_{ij}, but the matrix elements are not assumed to be Gaussian anymore (their distribution can be arbitrary but should decay fast enough).

In the large N limit μ_N converges to the so-called Wigner semi-circle law, an absolutely continuous probability measure supported on $[-2, 2]$ with the density

$$\rho_{\rm sc}(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2} \ . \tag{1}$$

This is true for all Wigner matrices, independently of the precise form of the distribution of the matrix entries, another example of the universality phenomenon.

In the following we shall sketch two proofs of this result, one specific for the GOE ensemble and a more general one.

1 Preamble on Gaussian variables

Consider an invertible $N \times N$ matrix A, assumed for simplicity to be definite positive. Using a Gaussian integral derive the following identity,

$$(A^{-1})_{ii} = \frac{1}{A_{ii} - \sum\limits_{\substack{j,k=1 \\ j \neq i, k \neq i}}^{N} A_{ij} ((A^{(i)})^{-1})_{jk} A_{ki}} \ , \tag{2}$$

where $A^{(i)}$ denotes the matrix of size $N-1$ obtained from A by removing the i-th row and column of A. This is a special case of the so-called Schur's complement lemma for the inverse of block matrices.

We recall that if X is a Gaussian random variable with zero mean and F a function,

$$\mathbb{E}[X F(X)] = \mathbb{E}\left[X^2\right] \ \mathbb{E}[F'(X)] \ . \tag{3}$$

This identity can be generalized to the case of a vector of n Gaussian random variables, $(X_1, \dots, X_n)$, each one of zero mean, with a function F of n variables,

$$\mathbb{E}[X_i F(X_1, \dots, X_n)] = \sum_{j=1}^{n} \mathbb{E}[X_i X_j] \ \mathbb{E}\left[\frac{\partial F}{\partial X_j}(X_1, \dots, X_n)\right] \ . \tag{4}$$

2 The semi-circle law for GOE random matrices

In this part of the problem M is a $N \times N$ matrix drawn from the GOE ensemble. We define the resolvent matrix of M as $G(z) = (M - z\,\mathbb{I})^{-1}$, where $\mathbb{I}$ is the $N \times N$ identity matrix, and z a complex number with $\mathrm{Im}\, z > 0$, and denote $g_N(z) = \frac{1}{N}\mathrm{Tr}\ G(z)$ its normalized trace.

1. Explain why
$$\mu_N(\lambda) = \frac{1}{\pi} \lim_{\eta \to 0^+} \mathrm{Im}\ g_N(\lambda + i\,\eta) \ . \tag{5}$$
2. Show that for an invertible matrix A one has
$$\frac{\partial (A^{-1})_{ij}}{\partial A_{kl}} = -(A^{-1})_{ik}(A^{-1})_{lj} \ . \tag{6}$$
3. Using the Gaussian formula (4) compute the value of $\mathbb{E}\left[G_{ij}(z)M_{kl}\right]$, distinguishing the cases $k = l$ and $k \neq l$.
4. Deduce from this result that
$$\mathbb{E}\left[\mathrm{Tr}\left(G(z)M\right)\right] = -\frac{1}{N}\mathbb{E}\left[(\mathrm{Tr}\, G(z))^2\right] - \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left(G(z)^2\right)\right] \ . \tag{7}$$
5. Simplify this expression to obtain
$$\mathbb{E}[g_N(z)^2] + z\,\mathbb{E}[g_N(z)] + 1 = -\frac{1}{N^2}\mathbb{E}\left[\mathrm{Tr}\left(G(z)^2\right)\right] \ . \tag{8}$$
6. Argue that the right hand side of this equation is negligible in the large N limit. We shall furthermore admit that in this limit the random variable $g_N(z)$ concentrates around its average, whose limit will be denoted $g(z)$. Write the equation satisfied by $g(z)$, solve it, and conclude to obtain the semi-circle law of Eq. (1).

3 The sketch of a proof by recursion for Wigner matrices

We consider now that M is a Wigner random matrix of a large size N.

1. Justify the following equation for the resolvent matrix :
$$G_{11}(z) = \frac{1}{M_{11} - z - \sum\limits_{j,k=2}^{N} M_{1j}\widetilde{G}_{jk}(z)M_{k1}} \ , \tag{9}$$
where $\widetilde{G}$ is the resolvent for the $(N-1) \times (N-1)$ matrix $\widetilde{M}$ obtained from M by removing its first line and column.
2. Deduce from this result that
$$\mathbb{E}\left[\frac{1}{G_{11}(z)}\right] = -z - \frac{1}{N}\,\mathbb{E}[\mathrm{Tr}\,\widetilde{G}(z)] \ . \tag{10}$$
3. In the large N limit one can prove the concentration of $G(z)$ around $g(z)\mathbb{I}$, where $g(z)$ is the limit of $\mathbb{E}[g_N(z)]$. Admitting this result show that (10) implies the universality of the semi-circle law for Wigner matrices.